A_∞-morphisms with several entries

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March 4, 2013

Abstract

We show that morphisms from n A_∞-algebras to a single one are maps over an operad module with $n+1$ commuting actions of the operad A_∞, whose algebras are conventional A_∞-algebras. Similar statement holds for homotopy unital A_∞-algebras. The operad A_∞ and modules over it have two useful gradings related by isomorphisms which change the degree. The composition of A_∞-morphisms with several entries is presented as a convolution of a coalgebra-like and an algebra-like structures. For this sake notions of lax $\mathcal{C}at$-span multicategories and multifunctors are introduced. They are lax versions of strict multicategories and multifunctors associated with the monad of free strict monoidal category.

It is well-known that operads play a prominent part in the study of A_∞-algebras. In particular, A_∞-algebras in the conventional sense [Sta63] are algebras over the **dg**-operad A_∞, a resolution (a cofibrant replacement) of the **dg**-operad As of associative non-unital **dg**-algebras. What about morphisms $A \to B$ of A_∞-algebras? It is shown in [Lyu11] that they are maps over certain bimodule over the **dg**-operad A_∞. This bimodule is a resolution (a cofibrant replacement) of the corresponding As-bimodule.

The current article addresses morphisms with several arguments $f : A_1, \dots, A_n \to B$ of A_∞-algebras. We explain that they are maps over certain $n \wedge 1$-operad A_∞-module F_n. The latter means an $\mathbb{N}^n$-graded complex with n left and one right pairwise commuting actions of A_∞. Furthermore, it is a resolution (a cofibrant replacement) of the corresponding notion for associative **dg**-algebras without unit.

The unital case is quite similar to the non-unital one. There is an operad A_∞^{hu} governing homotopy unital A_∞-algebras. Homotopy unital morphisms $A \to B$ are controlled by an operad A_∞^{hu}-bimodule [Lyu11]. In the current article we describe the $n \wedge 1$-operad A_∞^{hu}-module F_n^{hu} responsible for homotopy unital A_∞-morphisms $f : A_1, \dots, A_n \to B$. We see that it is a resolution (a cofibrant replacement) of the corresponding $n \wedge 1$-operad module over the operad of associative unital **dg**-algebras.

The **dg**-operad of A_∞-algebras has two useful forms. The first, A_∞, is already presented as a resolution of the operad As. The second, A_∞, is easy to remember, because all generators have the same degree 1 and the expression for the differential contains no oscillating signs. These two are related by an isomorphism of operads that changes the

degrees in a prescribed way. Structure equations for this isomorphism use certain signs. These signs reappear in the formula for the differential in the first operad, A_∞. There are similar duplicates of other considered operads and modules over them: F_n, A_∞^{hu}, F_n^{hu}, etc.

The definition and main properties of $n \wedge 1$-operad modules pop out in the study of lax $\mathcal{C}at$-span multicategories – one more direction treated in the article as a category theory base of the whole subject. These lax multicategories generalize strict multicategories associated with the monad of free strict monoidal category. Composition of A_∞- and A_∞^{hu}-morphisms of several arguments is presented as convolution of a certain colax $\mathcal{C}at$-span multifunctor viewed as a coalgebra and the lax $\mathcal{C}at$-span multifunctor $\mathcal{H}om$ viewed as an algebra. This gives the multicategory of A_∞- or A_∞^{hu}-algebras.

0.1. Notations and conventions. We denote by $\mathbb{N}$ the set of non-negative integers $\mathbb{Z}_{\geqslant 0}$. By norm on $\mathbb{N}^n$ we mean the function $|\cdot| : \mathbb{N}^n \to \mathbb{N}$, $j \mapsto |j| = \sum_{i=1}^n j^i$. Let $\Bbbk$ denote the ground commutative ring. Tensor product $\otimes_\Bbbk$ will be denoted simply $\otimes$. When a $\Bbbk$-linear map f is applied to an element x, the result is typically written as $x.f = xf$. The tensor product of two maps of graded $\Bbbk$-modules f, g of certain degree is defined so that for elements x, y of arbitrary degree

$$(x \otimes y).(f \otimes g) = (-1)^{\deg y \cdot \deg f} x.f \otimes y.g.$$

In other words, we strictly follow the Koszul rule. Composition of $\Bbbk$-linear maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ is usually denoted $f \cdot g = fg : X \to Z$. For other types of maps composition is often written as $g \circ f = gf$.

We assume that each set is an element of some universe. This universe is not fixed through the whole article. For instance, the category of categories $\mathcal{C}at$ means the category of $\mathscr{U}'$-small, locally $\mathscr{U}$-small categories for some universes $\mathscr{U} \in \mathscr{U}'$ (thus, a category $\mathcal{C}$ is in $\mathcal{C}at$ iff $\operatorname{Ob}\mathcal{C} \in \mathscr{U}'$ and $\mathcal{C}(X,Y) \in \mathscr{U}$ for all $X, Y \in \operatorname{Ob}\mathcal{C}$). These universes are used tacitly, without being explicitly mentioned.

We consider the category of totally ordered finite sets and their non-decreasing maps. An arbitrary totally ordered finite set is isomorphic to a unique set $\mathbf{n} = \{1 < 2 < \dots < n\}$ via a unique isomorphism, $n \geqslant 0$. Functions of totally ordered finite set that we use in this article are assumed to *depend only on the isomorphism class of the set*. Thus, it suffices to define them only for skeletal totally ordered finite sets $\mathbf{n}$. The full subcategory of such sets and their non-decreasing maps is denoted $\mathcal{O}_{\mathsf{sk}}$. The full subcategory of $\mathcal{S}et$ formed by $\mathbf{n}$, $n \geqslant 0$, is denoted $\mathcal{S}_{\mathsf{sk}}$.

Whenever $I \in \operatorname{Ob}\mathcal{O}_{\mathsf{sk}}$, there is another totally ordered set $[I] = \{0\} \sqcup I$ containing I, where element 0 is the smallest one. Thus, $[n] = [\mathbf{n}] = \{0 < 1 < 2 < \dots < n\}$.

The list $A, \dots, A$ consisting of n copies of the same object A is denoted ${}^nA$.

For any graded $\Bbbk$-module M denote by $sM = M[1]$ the same module with the grading shifted by 1: $M[1]^k = M^{k+1}$. Denote by $\sigma : M \to M[1]$, $M^k \ni x \mapsto x \in M[1]^{k-1}$ the "identity map" of degree $\deg\sigma = -1$.

0.2. Motivation. A_∞-algebras and A_∞-categories arise in symplectic topology in the studies of Floer cohomology of Lagrangian submanifolds of symplectic manifolds, see the monograph of K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono [FOOO09]. This article is devoted to A_∞-algebras, which in particular are cochain complexes of $\Bbbk$-modules (differential graded $\Bbbk$-modules)

$$\dots \xrightarrow{d} X^{-1} \xrightarrow{d} X^0 \xrightarrow{d} X^1 \xrightarrow{d} \dots, \quad d^2 = 0.$$

Denote by $(\mathbf{dg}, \otimes_\Bbbk)$ the monoidal category of cochain complexes of $\Bbbk$-modules with chain maps as morphisms. Tensor product of complexes is denoted $\otimes = \otimes_\Bbbk$ as usual. Sometimes we denote the same product by $\boxtimes$ instead. The reason lies in expected generalization from A_∞-algebras to A_∞-categories. The latter have the underlying structure of a $\mathbf{dg}$-quiver. Unlike $\mathbf{dg}$-modules $\mathbf{dg}$-quivers $\mathcal{A}$ and $\mathcal{B}$ admit two products: the external product $\mathcal{A} \boxtimes \mathcal{B}$ with the set of objects $\operatorname{Ob} \mathcal{A} \boxtimes \mathcal{B} = \operatorname{Ob} \mathcal{A} \times \operatorname{Ob} \mathcal{B}$ and the tensor product $\mathcal{A} \otimes \mathcal{B}$ defined if and only if $\operatorname{Ob} \mathcal{A} = \operatorname{Ob} \mathcal{B}$. In the latter case $\operatorname{Ob} \mathcal{A} \otimes \mathcal{B} = \operatorname{Ob} \mathcal{A}$. For $\mathbf{dg}$-modules viewed as a particular case of $\mathbf{dg}$-quivers ($\operatorname{Ob} \mathcal{A} = \operatorname{Ob} \mathcal{B} = \{1\}$) both products coincide.

A$_\infty$-algebras are complexes $A \in \mathbf{dg}$ with the differential m_1 and operations $m_n : A^{\otimes n} \to A$, $\deg m_n = 2 - n$, for $n \geqslant 2$ such that

$$\sum_{j+p+q=n} (-1)^{jp+q} (1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}) \cdot m_{j+1+q} = 0 \tag{0.1}$$

for all $n \geqslant 2$. For instance, binary multiplication m_2 is a chain map, it is associative up to the boundary of the homotopy m_3:

$$(m_2 \otimes 1)m_2 - (1 \otimes m_2)m_2 = m_3 m_1 + (1 \otimes 1 \otimes m_1 + 1 \otimes m_1 \otimes 1 + m_1 \otimes 1 \otimes 1)m_3,$$

and so on.

0.3. dg-operads. Informally, non-symmetric operads are collections of operations, which can be performed without permuting the arguments, in algebras of a certain type. In this article an operad will mean a *non-symmetric* operad.

A *(non-symmetric) operad* $\mathcal{O}$ is a collection of sets $(\mathcal{O}(n))_{n \in \mathbb{N}}$ – operations of arity n, an associative family of substitution compositions

$$\mu : \mathcal{O}(n_1) \times \dots \times \mathcal{O}(n_k) \times \mathcal{O}(k) \to \mathcal{O}(n_1 + \dots + n_k)$$

(one for each $\Bbbk$-tuple $(n_1, \dots, n_k) \in \mathbb{N}^k$, $k \in \mathbb{N}$), which has a two-sided unit $\eta \in \mathcal{O}(1)$ – the identity operation.

0.4 Example. The operad *as* of semigroups without unit has precisely one operation $m^{(n)} : (x_1, \dots, x_n) \mapsto x_1 \dots x_n$ for each $n > 0$. Thus, $as(n) = \{m^{(n)}\}$ for positive n and $as(0) = \varnothing$.

Similarly, there is the operad *as1* in $\mathcal{S}et$ with $as1(n) = \{m^{(n)}\}$ for all $n \geqslant 0$. *as1*-algebras are monoids – semigroups with a unit,which is implemented by the nullary operation $m^{(0)}$.

Operations from a **dg**-operad have in addition a degree and a boundary. Category $\mathcal{M} = \mathbf{dg}^{\mathbb{N}}$ of collections of complexes $(\mathcal{W}(n))_{n\in\mathbb{N}}$ is equipped with the tensor product $\odot$:

$$(\mathcal{U} \odot \mathcal{W})(n) = \bigoplus_{n_1+\dots+n_k=n}^{k\geqslant 0} \mathcal{U}(n_1) \otimes \dots \otimes \mathcal{U}(n_k) \otimes \mathcal{W}(k).$$

The unit object $\mathbb{1}$ has $\mathbb{1}(1) = \Bbbk$, $\mathbb{1}(n) = 0$ for $n \neq 1$.

A *(non-symmetric)* **dg**-*operad* $\mathcal{O}$ is a monoid in $\mathcal{M} = (\mathbf{dg}^{\mathbb{N}}, \odot)$, say $(\mathcal{O}, \mu : \mathcal{O} \odot \mathcal{O} \to \mathcal{O}, \eta : \mathbb{1} \to \mathcal{O})$. Multiplication consists of maps

$$\mu : \mathcal{O}(n_1) \otimes \dots \otimes \mathcal{O}(n_k) \otimes \mathcal{O}(k) \to \mathcal{O}(n_1 + \dots + n_k). \tag{0.2}$$

0.5 Example. For any complex $X \in \mathbf{dg}$ there is the **dg**-operad $\mathcal{E}nd\, X$ of its endomorphisms. It has $(\mathcal{E}nd\, X)(n) = \underline{\mathbf{dg}}(X^{\otimes n}, X)$, the complex of $\Bbbk$-linear maps $X^{\otimes n} \to X$ of certain degree. Here $\underline{\mathbf{dg}}$ is the category enriched in **dg** due to closedness of $(\mathbf{dg}, \otimes)$.

0.6 Definition. An algebra X over a **dg**-operad $\mathcal{O}$ is a complex X together with a morphism of operads $\mathcal{O} \to \mathcal{E}nd\, X$ (morphism of monoids in $\mathcal{M}$).

0.7 Example. The **dg**-operad As is the $\Bbbk$-linear envelope of the operad *as* in $\mathcal{S}et$. They have $As(0) = 0$ and $As(n) = \Bbbk m^{(n)} = \Bbbk$ for $n > 0$. The identity operation $m^{(1)}$ is the unit of the operad, and $m^{(2)} = m$ is the binary multiplication. As-algebras are associative differential graded $\Bbbk$-algebras without unit.

Similarly, the operad *as1* in $\mathcal{S}et$ has the $\Bbbk$-linear envelope – the **dg**-operad $As1$ with $As1(n) = \Bbbk m^{(n)} = \Bbbk$ for all $n \geqslant 0$. Clearly, $As1$-algebras are associative differential graded $\Bbbk$-algebras with the multiplication $m^{(2)}$ and the unit $m^{(0)} = 1^{\mathsf{su}}$.

0.8. Model category structures. The following theorem is proved by Hinich in [Hin97, Section 2.2], except that he relates a category with the category of complexes **dg**, not with its power $\mathbf{dg}^S$. A generalization is given in [Lyu12, Theorem 1.2]. It has the same formulation as below, however, **dg** means there the category of differential graded modules over a graded commutative ring.

0.9 Theorem ([Hin97, Lyu12])**.** *Suppose that S is a set, a category $\mathcal{C}$ is complete and cocomplete and $F : \mathbf{dg}^S \rightleftarrows \mathcal{C} : U$ is an adjunction. Assume that U preserves filtering colimits. For any $x \in S$, $p \in \mathbb{Z}$ consider the object $\mathbb{K}[-p]_x$ of $\mathbf{dg}^S$, $\mathbb{K}[-p]_x(x) = \big(0 \to \Bbbk \xrightarrow{1} \Bbbk \to 0\big)$ (concentrated in degrees p and $p+1$), $\mathbb{K}[-p]_x(y) = 0$ for $y \neq x$. Assume that the chain map $U(\mathrm{in}_2) : UA \to U(F(\mathbb{K}[-p]_x) \sqcup A)$ is a quasi-isomorphism for all objects A of $\mathcal{C}$ and all $x \in S$, $p \in \mathbb{Z}$. Equip $\mathcal{C}$ with the classes of weak equivalences (resp. fibrations) consisting of morphisms f of $\mathcal{C}$ such that Uf is a quasi-isomorphism (resp. an epimorphism). Then the category $\mathcal{C}$ is a model category.*

We shall recall also several constructions used in the proof of this theorem. They describe cofibrations and trivial cofibrations in $\mathcal{C}$. Assume that $M \in \operatorname{Ob} \mathbf{dg}^S$, $A \in \operatorname{Ob} \mathcal{C}$,

$\alpha : M \to UA \in \mathbf{dg}^S$. Denote by $C = \operatorname{Cone}\alpha = (M[1] \oplus UA, d_{\mathrm{Cone}}) \in \operatorname{Ob}\mathbf{dg}^S$ the cone taken pointwise, that is, for any $x \in S$ the complex $C(x) = \operatorname{Cone}\bigl(\alpha(x) : M(x) \to (UA)(x)\bigr)$ is the usual cone. Denote by $\bar\imath : UA \to C$ the obvious embedding. Let $\varepsilon : FU(A) \to A$ be the adjunction counit. Following Hinich [Hin97, Section 2.2.2] define an object $A\langle M, \alpha\rangle \in \operatorname{Ob}\mathcal{C}$ as the pushout

$$\begin{tikzcd} FU(A) \arrow[r, "\varepsilon"] \arrow[d, "F\bar\imath"'] & A \arrow[d, "\bar\jmath"] \\ FC \arrow[r, "g"] & A\langle M, \alpha\rangle \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{0.3}$$

If $\alpha = 0$, then $A\langle M, 0\rangle \simeq F(M[1]) \sqcup A$ and $\bar\jmath = \mathrm{in}_2$ is the canonical embedding. We say that M consists of free $\Bbbk$-modules if for any $x \in S$, $p \in \mathbb{Z}$ the $\Bbbk$-module $M(x)^p$ is free.

The proof contains the following important statements. If M consists of free $\Bbbk$-modules and $d_M = 0$, then $\bar\jmath : A \to A\langle M, \alpha\rangle$ is a cofibration. It might be called an *elementary standard cofibration*. If

$$A \to A_1 \to A_2 \to \cdots$$

is a sequence of elementary standard cofibrations, B is a colimit of this diagram, then the "infinite composition" map $A \to B$ is a cofibration called a *standard cofibration* [Hin97, Section 2.2.3].

Assume that $N \in \operatorname{Ob}\mathbf{dg}^S$ consists of free $\Bbbk$-modules, $d_N = 0$ and $M = \operatorname{Cone}\bigl(1_{N[-1]}\bigr) = (N \oplus N[-1], d_{\mathrm{Cone}})$. Then for any morphism $\alpha : M \to UA \in \mathbf{dg}^S$ the morphism $\bar\jmath : A \to A\langle M, \alpha\rangle$ is a trivial cofibration in $\mathcal{C}$ and a standard cofibration, composition of two elementary standard cofibrations. It is called a *standard trivial cofibration*. Any (trivial) cofibration is a retract of a standard (trivial) cofibration [Hin97, Remark 2.2.5].

When $F : \mathbf{dg}^S \to \mathcal{C}$ is the functor of constructing a free $\mathbf{dg}$-algebra of some kind, the maps $\bar\jmath$ are interpreted as "adding variables to kill cycles".

The category Op of operads admits an adjunction $F : \mathbf{dg}^{\mathbb{N}} \rightleftarrows \mathrm{Op} : U$. Applying Theorem 0.9 to this category one gets [Lyu11, Proposition 1.8]

0.10 Proposition. *Define weak equivalences (resp. fibrations) in* Op *as morphisms f of* Op *such that Uf is a quasi-isomorphism (resp. an epimorphism). These classes make* Op *into a model category.*

This statement was proven previously in [Hin97], [Spi01, Remark 2] and follows from [Mur11, Theorem 1.1].

0.11 Example. Using Stasheff's associahedra one proves that there is a cofibrant replacement $\mathrm{A}_\infty \to As$ where the graded operad A_∞ is freely generated by n-ary operations m_n of degree $2 - n$ for $n \geqslant 2$. The differential is found as

$$m_n\partial = -\sum_{j+p+q=n}^{1<p<n} (-)^{jp+q}(1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}) \cdot m_{j+1+q}. \tag{0.4}$$

Basis $(m(t))$ of $\mathrm{A}_\infty = T\big(\Bbbk\{m_n \mid n \geqslant 2\}\big)$ over $\Bbbk$ is indexed by isomorphism classes of ordered rooted trees t without unary vertices (those with one incoming edge). The tree t^1 which has just one vertex (the root and the leaf) corresponds to the unit from $\mathrm{A}_\infty(1)$.

Algebras over the **dg**-operad A_∞ are precisely A_∞-algebras in the sense of (0.1).

Furthermore, the chain map $\mathrm{A}_\infty(n) \to As(n)$ is homotopy invertible for each $n \geqslant 1$. One way to prove it is implied by a remark of Markl [Mar96, Example 4.8]. Another proof uses the operad of Stasheff associahedra [Sta63] and the configuration space of $(n+1)$-tuples of points on a circle considered by Seidel in his book [Sei08]. Details can be found in [BLM08, Proposition 1.19].

0.12. Morphisms of operads. Besides usual homomorphisms of operads, which are chain maps of degree 0, we consider also maps that change the degree.

0.13 Definition. A **dg**-operad homomorphism $t : \mathcal{O} \to \mathcal{P}$ of degree $\bar{t} = r \in \mathbb{Z}$ is a collection of homogeneous $\Bbbk$-linear maps $t(n) : \mathcal{O}(n) \to \mathcal{P}(n)$, $n \geqslant 0$, of degree $g(n) = (1-n)r$ such that

- $1_{\mathcal{O}}.t(1) = 1_{\mathcal{P}}$;
- for all $k, n_1, \dots, n_k \in \mathbb{N}$ the following square commutes up to a certain sign:

$$\begin{array}{ccc} \mathcal{O}(n_1)\otimes\cdots\otimes\mathcal{O}(n_k)\otimes\mathcal{O}(k) & \xrightarrow{\mu} & \mathcal{O}(n_1+\cdots+n_k) \\ {\scriptstyle t(n_1)\otimes\cdots\otimes t(n_k)\otimes t(k)}\big\downarrow & (-1)^c & \big\downarrow{\scriptstyle t(n_1+\cdots+n_k)} \\ \mathcal{P}(n_1)\otimes\cdots\otimes\mathcal{P}(n_k)\otimes\mathcal{P}(k) & \xrightarrow{\mu} & \mathcal{P}(n_1+\cdots+n_k) \end{array} \tag{0.5}$$

  where the tensor product of homogeneous right maps $t(\_)$ is that of the **dg**-category $\underline{\mathbf{dg}}$ and the sign is determined by

$$c = r\sum_{i=1}^{k}(i-1)(1-n_i) + \frac{r(r-1)}{2}\sum_{1\leqslant i<j\leqslant k}(1-n_i)(1-n_j) + \frac{r(r-1)}{2}(1-k)\sum_{i=1}^{k}(1-n_i); \tag{0.6}$$

- for all $n \in \mathbb{Z}$

$$d\cdot t(n) = (-1)^{g(n)}t(n)\cdot d : \mathcal{O}(n) \to \mathcal{P}(n).$$

Notice that the only functions $g : \mathbb{N} \to \mathbb{Z}$ that satisfy the equations

$$g(1) = 0, \qquad g(n_1)+\cdots+g(n_k)+g(k) = g(n_1+\cdots+n_k)$$

are functions $g(n) = (1-n)r$ for some $r \in \mathbb{Z}$.

0.14 Example. Let (X, d_X) be a complex of $\Bbbk$-modules and let $(X[1], d_{X[1]} = -\sigma^{-1}\cdot d_X\cdot\sigma)$ be its suspension. There is a **dg**-operad morphism

$$\Sigma = \mathcal{H}om(\sigma;\sigma^{-1}) = \mathcal{H}om(\sigma;1)\cdot\mathcal{H}om(1;\sigma^{-1}) : \mathcal{E}nd(X[1]) \to \mathcal{E}nd\,X$$

of degree 1. Thus, the mapping $f \mapsto (-1)^{nf}\sigma^{\otimes n} \cdot f \cdot \sigma^{-1}$,

$$\varSigma(n) = \underline{\mathbf{dg}}(\sigma^{\otimes n};1) \cdot \underline{\mathbf{dg}}(1;\sigma^{-1}) : \underline{\mathbf{dg}}(X[1]^{\otimes n}, X[1]) \to \underline{\mathbf{dg}}(X^{\otimes n}, X),$$

has degree $1-n$. The sign $(-1)^c$, $c = \sum_{i=1}^k (i-1)(1-n_i)$, pops out in the following procedure. Write down the tensor product corresponding to the left vertical arrow of (0.5) for $t = \varSigma$:

$$(\sigma^{\otimes n_1} \otimes \sigma^{-1}) \otimes (\sigma^{\otimes n_2} \otimes \sigma^{-1}) \otimes \cdots \otimes (\sigma^{\otimes n_k} \otimes \sigma^{-1}) \otimes (\sigma^{\otimes k} \otimes \sigma^{-1});$$

move factors σ^{-1} using the Koszul rule to their respective opponents, factors σ of $\sigma^{\otimes k}$, in order to cancel them and to obtain finally $\sigma^{\otimes(n_1+\dots+n_k)} \otimes \sigma^{-1}$.

Maps $\varSigma(n)$ commute with the differential in the graded sense because their factors $\sigma^{\pm 1}$ do.

0.15 Remark. Summands $r(r-1)/2\sum_{1\leqslant i<j\leqslant k}(1-n_i)(1-n_j)+(1-k)r(r-1)/2\sum_{i=1}^k(1-n_i)$ of c make sure that the composition of two morphisms of operads $t : \mathcal{O} \to \mathcal{P}$ and $u : \mathcal{P} \to \mathcal{Q}$ of degree $\bar{t}$ and $\bar{u}$ respectively be an operad morphism of degree $\bar{t}+\bar{u}$. Furthermore, if all homogeneous maps $t(n) : \mathcal{O}(n) \to \mathcal{P}(n)$ for $t : \mathcal{O} \to \mathcal{P}$ are invertible, than there is an inverse morphism of operads $t^{-1} : \mathcal{P} \to \mathcal{O}$ of degree $-\bar{t}$ with $t^{-1}(n) = t(n)^{-1}$.

Let $\mathcal{O}$ be a **dg**-operad, $\mathcal{P}$ be a graded operad and $t : \mathcal{O} \to \mathcal{P}$ be an invertible graded operad homomorphism of degree r ($1_{\mathcal{O}}.t(1) = 1_{\mathcal{P}}$ and (0.5) holds). Then $\mathcal{P}$ has a unique differential d which turns it into a **dg**-operad and $t : \mathcal{O} \to \mathcal{P}$ into a **dg**-operad homomorphism of degree r.

0.16 Example. There is a version of the **dg**-operad A_∞ denoted A_∞. This is a **dg**-operad freely generated as a graded operad by n-ary operations b_n of degree 1 for $n \geqslant 2$. The differential is defined as

$$b_n\partial = -\sum_{j+p+q=n}^{1<p<n} (1^{\otimes j} \otimes b_p \otimes 1^{\otimes q}) \cdot b_{j+1+q}.$$

Comparing the differentials we find that these two operads are isomorphic via an isomorphism of degree 1

$$\varSigma : A_\infty \to \mathrm{A}_\infty, \qquad b_i \mapsto m_i.$$

In fact, due to (0.5)

$$\begin{aligned}
&[(1^{\otimes j} \otimes b_p \otimes 1^{\otimes q})b_{j+1+q}].\varSigma(j+p+q) \\
&= (-1)^{j(1-p)+1-p}[(1^{\otimes j} \otimes b_p \otimes 1^{\otimes q}).(\varSigma(1)^{\otimes j} \otimes \varSigma(p) \otimes \varSigma(1)^{\otimes q})] \cdot [b_{j+1+q}.\varSigma(j+1+q)] \\
&= (-1)^{(j+1)(1-p)}[(1^{\otimes j} \otimes m_p \otimes 1^{\otimes q})m_{j+1+q}].
\end{aligned}$$

Therefore,

$$
\begin{aligned}
m_n\partial &= (b_n.\Sigma(n))\partial = (-1)^{1-n}(b_n\partial).\Sigma(n)\\
&= (-1)^n \sum_{j+p+q=n}^{1<p<n} [(1^{\otimes j}\otimes b_p\otimes 1^{\otimes q})b_{j+1+q}].\Sigma(n)\\
&= (-1)^n \sum_{j+p+q=n}^{1<p<n} (-1)^{(j+1)(1-p)}(1^{\otimes j}\otimes m_p\otimes 1^{\otimes q})m_{j+1+q}\\
&= -\sum_{j+p+q=n}^{1<p<n} (-1)^{jp+q}(1^{\otimes j}\otimes m_p\otimes 1^{\otimes q})\cdot m_{j+1+q}
\end{aligned}
$$

which coincides with (0.4). This fixes the differential on A_∞ since m_n generate the graded operad. An easy lemma shows that it suffices to verify graded commutation of the differential and any operad homomorphism on generators. In particular, $\Sigma : A_\infty \to \mathrm{A}_\infty$ commutes with ∂ in the graded sense.

Knowing that A_∞ is homotopy isomorphic to its cohomology As, we conclude that A_∞ is homotopy isomorphic to its cohomology as well. There is an isomorphism of degree 1 between graded operads $\Sigma : H^\bullet(A_\infty) \to As$. Hence, $H^\bullet(A_\infty(n)) = \Bbbk[1-n]$ for $n \geqslant 1$.

For any algebra $A \in \mathbf{dg}$ over the $\mathbf{dg}$-operad A_∞ the $\mathbf{dg}$-module $A[1]$ becomes an algebra over the $\mathbf{dg}$-operad A_∞ so that the square of operad homomorphisms commutes:

$$
\begin{tikzcd}
A_\infty \arrow[r] \arrow[d, "\Sigma"'] & \mathcal{E}nd\, A[1] \arrow[d, "\mathcal{H}om(\sigma;\sigma^{-1})"] \\
\mathrm{A}_\infty \arrow[r] & \mathcal{E}nd\, A
\end{tikzcd}
\quad , \qquad (-1)^n\sigma^{\otimes n}\cdot b_n\cdot\sigma^{-1} = m_n : A^{\otimes n} \to A, \quad n \geqslant 1. \tag{0.7}
$$

Verification is straightforward.

0.17 Example. Approaching homotopy unital A_∞-algebras we start with strictly unital ones. They are governed by the operad $\mathrm{A}_\infty^{\mathsf{su}}$ generated over A_∞ by a nullary degree 0 cycle $\mathbf{1}^{\mathsf{su}}$ subject to the following relations:

$$
(1\otimes\mathbf{1}^{\mathsf{su}})m_2 = 1, \quad (\mathbf{1}^{\mathsf{su}}\otimes 1)m_2 = 1, \quad (1^{\otimes a}\otimes\mathbf{1}^{\mathsf{su}}\otimes 1^{\otimes c})m_{a+1+c} = 0 \;\text{ if }\; a+c>1.
$$

There is a standard trivial cofibration and a homotopy isomorphism $\mathrm{A}_\infty^{\mathsf{su}} \rightarrowtail\!\!\sim\!\!\to \mathrm{A}_\infty^{\mathsf{su}}\langle\mathbf{1}^{\mathsf{su}} - \mathsf{i},\mathsf{j}\rangle = \mathrm{A}_\infty^{\mathsf{su}}\langle\mathsf{i},\mathsf{j}\rangle$, where i, j are two nullary operations, $\deg\mathsf{i} = 0$, $\deg\mathsf{j} = -1$, with $\mathsf{i}\partial = 0$, $\mathsf{j}\partial = \mathbf{1}^{\mathsf{su}} - \mathsf{i}$.

A cofibrant replacement $\mathrm{A}_\infty^{hu} \to As1$ is constructed as a $\mathbf{dg}$-suboperad of $\mathrm{A}_\infty^{\mathsf{su}}\langle\mathsf{i},\mathsf{j}\rangle$ generated as a graded operad by i and n-ary operations of degree $4-n-2k$

$$
m_{n_1;n_2;\dots;n_k} = (1^{\otimes n_1}\otimes\mathsf{j}\otimes 1^{\otimes n_2}\otimes\mathsf{j}\otimes\dots\otimes 1^{\otimes n_{k-1}}\otimes\mathsf{j}\otimes 1^{\otimes n_k})m_{n+k-1},
$$

where $n=\sum_{q=1}^k n_q$, $k\geqslant 1$, $n_q\geqslant 0$, $n+k\geqslant 3$. Notice that the graded operad A_∞^{hu} is free. See [Lyu11, Section 1.11] for the proofs.

One can perform all the above steps also for the operad A_∞:

1) Adding to A_∞ a nullary degree -1 cycle $\mathbf{1}^{\mathsf{su}}$ subject to the relations:

$$(1\otimes\mathbf{1}^{\mathsf{su}})b_2=1,\quad (\mathbf{1}^{\mathsf{su}}\otimes 1)b_2=-1,\quad (1^{\otimes a}\otimes\mathbf{1}^{\mathsf{su}}\otimes 1^{\otimes c})b_{a+1+c}=0 \;\text{ if }\; a+c>1. \tag{0.8}$$

The resulting operad is denoted A_∞^{su}.

2) Adding to A_∞^{su} two nullary operations $\mathbf{i}$, $\mathbf{j}$, $\deg\mathbf{i}=-1$, $\deg\mathbf{j}=-2$, with $\mathbf{i}\partial=0$, $\mathbf{j}\partial=\mathbf{i}-\mathbf{1}^{\mathsf{su}}$. The standard trivial cofibration $A_\infty^{\mathsf{su}}\rightarrowtail\!\!\sim\!\!\rightarrow A_\infty^{\mathsf{su}}\langle\mathbf{i},\mathbf{j}\rangle$ is a homotopy isomorphism.

3) A_∞^{hu} is a $\mathbf{dg}$-suboperad of $A_\infty^{\mathsf{su}}\langle\mathbf{i},\mathbf{j}\rangle$ generated as a graded operad by $\mathbf{i}$ and n-ary operations of degree $3-2k$

$$b_{n_1;n_2;\dots;n_k}=(1^{\otimes n_1}\otimes\mathbf{j}\otimes 1^{\otimes n_2}\otimes\mathbf{j}\otimes\dots\otimes 1^{\otimes n_{k-1}}\otimes\mathbf{j}\otimes 1^{\otimes n_k})b_{n+k-1},$$

where $n=\sum_{q=1}^k n_q$, $k\geqslant 1$, $n_q\geqslant 0$, $n+k\geqslant 3$.

The obtained operads are related to the previous ones by invertible homomorphisms of degree 1, extending $\varSigma: b_n\mapsto m_n$,

$$\varSigma: A_\infty^{\mathsf{su}}\to\mathrm{A}_\infty^{\mathsf{su}},\ \mathbf{1}^{\mathsf{su}}\mapsto\mathsf{1}^{\mathsf{su}};\quad \varSigma: A_\infty^{\mathsf{su}}\langle\mathbf{i},\mathbf{j}\rangle\to\mathrm{A}_\infty^{\mathsf{su}}\langle\mathsf{i},\mathsf{j}\rangle,\ \mathbf{i}\mapsto\mathsf{i},\ \mathbf{j}\mapsto\mathsf{j};\quad \varSigma: A_\infty^{hu}\to\mathrm{A}_\infty^{hu}.$$

The latter is a restriction of the previous map. For algebras A over operads $\mathrm{A}_\infty^{\mathsf{su}}$, $\mathrm{A}_\infty^{\mathsf{su}}\langle\mathsf{i},\mathsf{j}\rangle$, A_∞^{hu} the complex $A[1]$ obtains a structure of an algebra over the operad A_∞^{su}, $A_\infty^{\mathsf{su}}\langle\mathbf{i},\mathbf{j}\rangle$ or A_∞^{hu} due to a property similar to (0.7), in particular,

$$\mathbf{1}^{\mathsf{su}}\sigma^{-1}=\mathsf{1}^{\mathsf{su}},\qquad \mathbf{i}\sigma^{-1}=\mathsf{i},\qquad \mathbf{j}\sigma^{-1}=\mathsf{j}:\Bbbk\to A.$$

0.18. Morphisms of A_∞-algebras. Objects of the category $\mathsf{C}_\Bbbk^{\mathbb{N}\sqcup\mathbb{N}\sqcup\mathbb{N}}$ are written as triples of collections $(\mathcal{A};\mathcal{P};\mathcal{B})=(\mathcal{A}(n);\mathcal{P}(n);\mathcal{B}(n))_{n\in\mathbb{N}}$ of complexes. An *operad bimodule* is defined as a triple $(\mathcal{A};\mathcal{P};\mathcal{B})$, consisting of operads $\mathcal{A}$, $\mathcal{B}$ and an $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{P}$ in the monoidal category $(\mathbf{dg}^{\mathbb{N}},\odot)$. The actions $\lambda:\mathcal{A}\odot\mathcal{P}\to\mathcal{P}$ and $\rho:\mathcal{P}\odot\mathcal{B}\to\mathcal{P}$ consist of chain maps

$$\lambda_{n_1,\dots,n_k}:\mathcal{A}(n_1)\otimes\dots\otimes\mathcal{A}(n_k)\otimes\mathcal{P}(k)\to\mathcal{P}(n_1+\dots+n_k),$$
$$\rho_{n_1,\dots,n_k}:\mathcal{P}(n_1)\otimes\dots\otimes\mathcal{P}(n_k)\otimes\mathcal{B}(k)\to\mathcal{P}(n_1+\dots+n_k).$$

The category of operad bimodules ${}_1\mathrm{Op}_1$ has morphisms $(f;h;g):(\mathcal{A};\mathcal{P};\mathcal{B})\to(\mathcal{C};\mathcal{Q};\mathcal{D})$, where $f:\mathcal{A}\to\mathcal{C}$, $g:\mathcal{B}\to\mathcal{D}$ are morphisms of $\mathbf{dg}$-operads and $h:\mathcal{P}\to{}_f\mathcal{Q}_g$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule morphism, where ${}_f\mathcal{Q}_g=\mathcal{Q}$ obtains its $\mathcal{A}$-$\mathcal{B}$-bimodule structure via f, g.

0.19 Example. Let X, Y be objects of $\mathbf{dg}$ (complexes of $\Bbbk$-modules). Define a collection $\mathcal{H}om(X,Y)$ as $\mathcal{H}om(X,Y)(n)=\underline{\mathbf{dg}}(X^{\otimes n},Y)$. Substitution composition $\mathcal{H}om(X,Y)\odot\mathcal{H}om(Y,Z)\to\mathcal{H}om(X,Z)$ and obvious units $\mathbb{1}\to\mathcal{H}om(X,X)$ make the category of complexes enriched in the monoidal category $(\mathbf{dg}^{\mathbb{N}},\odot)$. In particular, $\mathcal{E}nd\,X=\mathcal{H}om(X,X)$ are algebras in $\mathbf{dg}^{\mathbb{N}}$ ($\mathbf{dg}$-operads). The collection $\mathcal{H}om(X,Y)$ is an $\mathcal{E}nd\,X$-$\mathcal{E}nd\,Y$-bimodule. The multiplication and the actions are induced by substitution composition.

In the nearest sections we use the shorthand $(\mathcal{O},\mathcal{P})$ for an operad $\mathcal{O}$-bimodule $(\mathcal{O};\mathcal{P};\mathcal{O})$.

0.19.1. Morphisms come from bimodules. Consider the operad bimodule (As, As), where the first term is an operad and the second term is a regular bimodule. One easily checks that a morphism of operad bimodules

$$(As; As; As) \to (\mathcal{E}nd\, X; \mathcal{H}om(X, Y); \mathcal{E}nd\, Y)$$

amounts to a morphism $f : X \to Y$ of associative differential graded $\Bbbk$-algebras without units.

There is a pair of adjoint functors $T : \mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}\sqcup\mathbb{N}} \rightleftarrows {}_1\mathrm{Op}_1 : U$, $T(\mathcal{U}; \mathcal{X}; \mathcal{W}) = (T\mathcal{U}; T\mathcal{U} \odot \mathcal{X} \odot T\mathcal{W}; T\mathcal{W})$. Applying Hinich's Theorem 0.9 we get

0.20 Proposition (Proposition 2.2 [Lyu11])**.** *Define weak equivalences (resp. fibrations) in ${}_1\mathrm{Op}_1$ as morphisms f of ${}_1\mathrm{Op}_1$ such that Uf is a quasi-isomorphism (resp. an epimorphism). These classes make ${}_1\mathrm{Op}_1$ into a model category.*

Cofibrant replacement of a **dg**-operad bimodule $(\mathcal{O}, \mathcal{P})$ is a trivial fibration $(\mathcal{A}, \mathcal{F}) \to (\mathcal{O}, \mathcal{P})$ (surjective mapping inducing isomorphism in cohomology) such that the only map $(\mathbb{1}, 0) \to (\mathcal{A}, \mathcal{F})$ is a cofibration in ${}_1\mathrm{Op}_1$, for instance, the graded operad bimodule $(\mathcal{A}, \mathcal{F})$ is freely generated.

0.21 Definition. A **dg**-operad bimodule homomorphism $(u, v, w) : (\mathcal{A}; \mathcal{P}; \mathcal{B}) \to (\mathcal{C}; \mathcal{Q}; \mathcal{D})$ of degree $r \in \mathbb{Z}$ is a pair of **dg**-operad homomorphisms $u : \mathcal{A} \to \mathcal{C}$, $w : \mathcal{B} \to \mathcal{D}$ of degree r and a collection of homogeneous $\Bbbk$-linear maps $v(n) : \mathcal{P}(n) \to \mathcal{Q}(n)$, $n \geqslant 0$, of degree $r(1 - n)$ such that

- for all $k, n_1, \dots, n_k \in \mathbb{N}$ the following squares commute up to the sign given in (0.6):

$$\begin{tikzcd}
\mathcal{A}(n_1) \otimes \cdots \otimes \mathcal{A}(n_k) \otimes \mathcal{P}(k) \arrow[r, "\lambda"] \arrow[d, "u(n_1)\otimes\cdots\otimes u(n_k)\otimes v(k)"'] \arrow[dr, phantom, "(-1)^c"] & \mathcal{P}(n_1 + \cdots + n_k) \arrow[d, "v(n_1+\cdots+n_k)"] \\
\mathcal{C}(n_1) \otimes \cdots \otimes \mathcal{C}(n_k) \otimes \mathcal{Q}(k) \arrow[r, "\lambda"] & \mathcal{Q}(n_1 + \cdots + n_k)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \otimes \mathcal{B}(k) \arrow[r, "\rho"] \arrow[d, "v(n_1)\otimes\cdots\otimes v(n_k)\otimes w(k)"'] \arrow[dr, phantom, "(-1)^c"] & \mathcal{P}(n_1 + \cdots + n_k) \arrow[d, "v(n_1+\cdots+n_k)"] \\
\mathcal{Q}(n_1) \otimes \cdots \otimes \mathcal{Q}(n_k) \otimes \mathcal{D}(k) \arrow[r, "\rho"] & \mathcal{Q}(n_1 + \cdots + n_k)
\end{tikzcd}$$

- for all $n \in \mathbb{N}$
$$d \cdot v(n) = (-1)^{r(1-n)} v(n) \cdot d : \mathcal{P}(n) \to \mathcal{Q}(n).$$

Properties of **dg**-operad bimodule homomorphisms are quite similar to those of **dg**-operad homomorphisms, described in Remark 0.15.

0.22 Proposition (cf. Proposition 2.7 [Lyu11])**.** *There is an operad bimodule (A_∞, F_1) freely generated by n-ary elements f_n of degree 0 over the graded operad A_∞. The differential for it is given by*

$$f_k\partial = \sum_{r+n+t=k}^{n>1} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) f_{r+1+t} - \sum_{i_1+\dots+i_l=k}^{l>1} (f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_l}) b_l. \tag{0.9}$$

F_1-maps are A_∞-algebra morphisms (for algebras written with operations b_n). There is an isomorphic form of this bimodule – the operad bimodule $(\mathrm{A}_\infty, \mathrm{F}_1)$ freely generated by n-ary elements f_n of degree $1-n$ over the graded operad A_∞. The differential for it is given by

$$\mathsf{f}_k\partial = \sum_{r+n+t=k}^{n>1} (-1)^{(r+1)n+t-1} (1^{\otimes r} \otimes m_n \otimes 1^{\otimes t}) \mathsf{f}_{r+1+t} - \sum_{i_1+\dots+i_l=k}^{l>1} (-1)^\sigma (\mathsf{f}_{i_1} \otimes \mathsf{f}_{i_2} \otimes \dots \otimes \mathsf{f}_{i_l}) m_l,$$

$$\sigma = k - 1 + \sum_{j=1}^{k} j(1-i_j).$$

The isomorphism between these bimodules

$$(\varSigma, \varSigma) : (A_\infty, F_1) \to (\mathrm{A}_\infty, \mathrm{F}_1), \qquad b_i \mapsto m_i, \quad f_k \mapsto \mathsf{f}_k$$

has degree 1. F_1-maps are A_∞-algebra morphisms $A \to B$ (for algebras written with operations m_n). The two notions of A_∞-morphisms agree in the sense that the square of operad bimodule maps

$$\begin{tikzcd}
(A_\infty; F_1; A_\infty) \arrow[r] \arrow[d, "(\varSigma;\varSigma;\varSigma)"'] & (\mathcal{E}nd\, A[1]; \mathcal{H}om(A[1], B[1]); \mathcal{E}nd\, B[1]) \arrow[d, "(\mathcal{H}om(\sigma;\sigma^{-1});\mathcal{H}om(\sigma;\sigma^{-1});\mathcal{H}om(\sigma;\sigma^{-1}))"] \\
(\mathrm{A}_\infty; \mathrm{F}_1; \mathrm{A}_\infty) \arrow[r] & (\mathcal{E}nd\, A; \mathcal{H}om(A, B); \mathcal{E}nd\, B)
\end{tikzcd}$$

commutes. The bimodule $(\mathrm{A}_\infty, \mathrm{F}_1)$ is a cofibrant replacement of (As, As). $(\mathrm{A}_\infty, \mathrm{F}_1) \to (As, As)$ is a homotopy isomorphism in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}}$.

Proof. There is a degree preserving **dg**-operad bimodule isomorphism $(\mathrm{A}_\infty, \mathrm{F}_1) \to (\mathrm{A}_\infty, \mathrm{F}'_1)$, $\mathsf{f}_k \mapsto (-1)^{1-k}\mathsf{f}'_k$, to the bimodule presented in [Lyu11], whose generators are denoted here by f'_k. Thus all previously proven properties are inherited by the A_∞-bimodule described above. □

0.23. Tensor coalgebra. The *tensor* $\Bbbk$*-module* of $A[1]$ is $T(A[1]) = \oplus_{n\geqslant 0} T^n(A[1]) = \oplus_{n\geqslant 0} A[1]^{\otimes n}$. Multiplication in an A_∞-algebra A is given by the operations of degree +1

$$b_n : T^n(A[1]) = A[1]^{\otimes n} \to A[1], \qquad n \geqslant 1.$$

Recall that $\Bbbk$-linear maps are composed *from left to right*. Operations b_n have to satisfy the A_∞-equations, $n \geqslant 1$:

$$\sum_{r+k+t=n} (1^{\otimes r} \otimes b_k \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^n(A[1]) \to A[1].$$

Tensor $\Bbbk$-module $T(A[1])$ has a coalgebra structure: the cut coproduct

$$\Delta(x_1 \otimes x_2 \otimes \dots \otimes x_n) = \sum_{k=0}^{n} x_1 \otimes \dots \otimes x_k \bigotimes x_{k+1} \otimes \dots \otimes x_n.$$

An A_∞-structure on a graded $\Bbbk$-module A is equivalent to $b^2 = 0$, where $b : T(A[1]) \to T(A[1])$ is a coderivation of degree $+1$ given by the formula

$$b = \sum_{r+k+t=n} 1^{\otimes r} \otimes b_k \otimes 1^{\otimes t} : T^n(A[1]) \to T(A[1]), \qquad b_0 = 0.$$

In particular, $b\Delta = \Delta(1 \otimes b + b \otimes 1)$.

0.24. Comultiplication and composition. In order to have an associative composition of $(\mathcal{O}, \mathcal{F})$-morphisms, we postulate an associative counital comultiplication $\Delta : \mathcal{F} \to \mathcal{F} \odot_{\mathcal{O}} \mathcal{F}$ in the category of $\mathcal{O}$-bimodules.

Suppose that A, B, C are $\mathcal{O}$-algebras and $g : \mathcal{F} \to \mathcal{H}om(A, B)$, $h : \mathcal{F} \to \mathcal{H}om(B, C)$ are $(\mathcal{O}, \mathcal{F})$-morphisms. Then their composition is defined as the convolution

$$g \cdot h = \bigl[\mathcal{F} \xrightarrow{\Delta} \mathcal{F} \odot_{\mathcal{O}} \mathcal{F} \xrightarrow{g \odot h} \mathcal{H}om(A, B) \odot_{\mathcal{E}nd\, B} \mathcal{H}om(B, C) \to \mathcal{H}om(A, C)\bigr].$$

For (As, As) the comultiplication is the identity map. For A_∞-morphisms the comultiplication is chosen as

$$\Delta : F_1 \to F_1 \odot_{A_\infty} F_1, \quad f_n\Delta = \sum_{i_1+\dots+i_k=n} (f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_k}) \otimes f_k,$$

see [Lyu11, Section 2.16], which results in the composition

$$(g \cdot h)_n = \sum_{i_1+\dots+i_k=n} (g_{i_1} \otimes g_{i_2} \otimes \dots \otimes g_{i_k}) h_k.$$

0.25. Homomorphisms with n arguments. A_∞-morphisms with several arguments $f : A_1, \dots, A_n \to B$ are defined as augmented **dg**-coalgebra morphisms

$$\hat{f} : T(A_1[1]) \otimes \dots \otimes T(A_n[1]) \to T(B[1]).$$

Here both augmented graded coalgebras $(C, \Delta, \varepsilon, \eta)$ are of the form $(\Bbbk \oplus \bar{C}, \Delta(x) = 1 \otimes x + x \otimes 1 + \bar{\Delta}(x)\ \forall\, x \in \bar{C}, \mathrm{pr}_1, \mathrm{in}_1)$, where the non-counital coassociative coalgebra $(\bar{C}, \bar{\Delta})$

is conilpotent (cocomplete [LH03, Section 1.1.2], [Kel06, Section 4.3]). Thus $(\bar{C}, \bar{\Delta})$ is identified with a $T^{\geqslant 1}$-coalgebra [BLM08, Proposition 6.8], see also Proposition 3.6 of the current article. In the category of such augmented graded coalgebras the target $\Bbbk \oplus T^{\geqslant 1}(B[1])$ is cofree, see Corollary 3.7, hence, augmented graded coalgebra morphisms $\hat{f}$ are in bijection with the degree 0 $\Bbbk$-linear maps

$$f : T(A_1[1]) \otimes \cdots \otimes T(A_n[1]) \to B[1],$$

whose restriction to $T^0(A_1[1]) \otimes \cdots \otimes T^0(A_n[1]) \simeq \Bbbk$ vanishes. The morphism $\hat{f}$ will be a chain map, $\hat{f}b = b\hat{f}$, if and only if

$$\begin{aligned}
&\sum_{q=1}^{n} \sum_{r+c+t=\ell^q}^{c>0} \big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{1^{\boxtimes(q-1)}\boxtimes(1^{\otimes r}\otimes b_c\otimes 1^{\otimes t})\boxtimes 1^{\boxtimes(n-q)}} \\
&T^{\ell^1} sA_1 \boxtimes \cdots \boxtimes T^{\ell^{q-1}} sA_{q-1} \boxtimes T^{r+1+t} sA_q \boxtimes T^{\ell^{q+1}} sA_{q+1} \boxtimes \cdots \boxtimes T^{\ell^n} sA_n \xrightarrow{f_{\ell-(c-1)e_q}} sB\big] \\
&= \sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0 \\ j_1+\dots+j_k=\ell}}^{k>0} \big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{\sim} \boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j_p^i} sA_i \\
&\qquad \xrightarrow{\sim} \otimes^{p\in\mathbf{k}} \boxtimes^{i\in\mathbf{n}} T^{j_p^i} sA_i \xrightarrow{\otimes^{p\in\mathbf{k}} f_{j_p}} \otimes^{p\in\mathbf{k}} sB \xrightarrow{b_k} sB\big]. \qquad (0.10)
\end{aligned}$$

0.26. Composition of A_∞-morphisms. A *tree* t is a composable sequence of non-decreasing maps of totally ordered finite sets $\mathbf{m} = \{1 < 2 < \cdots < m\}$ (objects of $\mathcal{O}_{\mathsf{sk}}$)

$$t = \big(t(0) \xrightarrow{t_1} t(1) \xrightarrow{t_2} \dots\ t(l-1) \xrightarrow{t_l} t(l) = \mathbf{1}\big). \qquad (0.11)$$

Composition of a family of A_∞-morphisms $(g_h^b)_{h>0}^{b\in t(h)}$ indexed by vertices of a tree t is

$$\begin{aligned}
g_j = \operatorname{comp}(t)(g_h^b)_j = \big[&\otimes^{a\in t(0)} T^{j^a} sA_0^a \xrightarrow{\otimes^{b_1\in t(1)}\widehat{g_1^{b_1}}} \otimes^{b_1\in t(1)} T^{j_1^{b_1}} sA_1^{b_1} \\
&\xrightarrow{\otimes^{b_2\in t(2)}\widehat{g_2^{b_2}}} \otimes^{b_2\in t(2)} T^{j_2^{b_2}} sA_2^{b_2} \to \dots \xrightarrow{g_l^1} sA_l^1\big],
\end{aligned}$$

and g_h^b is given via its components $g_{h,j}^b : \otimes^{a\in t_h^{-1}b} T^{j^a} sA_{h-1}^a \to sA_h^b$. Here j belongs to $\mathbb{N}^{t_h^{-1}b}$.

Explicit formula for the composition is

$$g_j = \sum_{\forall a\in t(0)\ |\tau(0,a)|=j^a}^{t-\text{tree } \tau} \otimes^{h\in I} \otimes^{b\in t(h)} \otimes^{p\in\tau(h,b)} g^b_{h,|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}}.$$

A t-tree is a functor $\tau : t \to \mathcal{O}_{\mathsf{sk}}$, $\tau(\mathrm{root}) = \mathbf{1}$, where the poset t is the free category (of paths) built on the quiver t oriented towards the root. It has the set of objects $\overline{\mathrm{v}}(t)$, the set of vertices of t, and the root is the terminal object of t. A *symmetric tree* is a sequence (0.11), where maps t_h are not supposed to be non-decreasing. A *braided tree* is

a symmetric tree (a functor) $t : [l] \to \mathcal{S}_{\mathsf{sk}}$ for which the maps t_h satisfy an extra condition: for all $0 \leqslant p < q < r \leqslant l$, $a, b \in t(p)$ inequalities $a < b$ and $t_{p\to q}(a) > t_{p\to q}(b)$ imply $t_{p\to r}(a) \geqslant t_{p\to r}(b)$ (see [BLM08, Definition 2.3]). Labelling of a (symmetric) tree in a set S consists of functions $\ell : t(h) \to S$, $0 \leqslant h \leqslant l$.

0.27. Graded complexes. Now let us explain what is new in the main part of the article. In Section 1 we define such notions as lax $\mathcal{C}at$-span multicategories and their particular cases: lax $\mathcal{C}at$-span operads, lax $\mathcal{C}at$-multicategories and lax $\mathcal{C}at$-operads. These are accompanied with definitions of lax $\mathcal{C}at$-span multifunctors and $\mathcal{C}at$-span multinatural transformations. Together they form a 2-category, whose objects are lax $\mathcal{C}at$-span multicategories.

0.27.1. $\mathcal{C}at$-operad of graded $\Bbbk$-modules. Let us describe examples G, DG of a weak $\mathcal{C}at$-operad: that of (differential) graded $\Bbbk$-modules. We define $\mathsf{G}(n) = \mathbf{gr}^{\mathbb{N}^n}$, $\mathsf{DG}(n) = \mathbf{dg}^{\mathbb{N}^n}$. The structure of one is obtained from the structure of the other by forgetting or introducing the differential. So we describe only one of them.

A functor is given for a tree t

$$\circledast(t) : \prod_{(h,b)\in \mathrm{v}(t)} \mathsf{G}(t_h^{-1}b) \to \mathsf{G}(t(0)), \qquad (\mathcal{P}_h^b)_{(h,b)\in\mathrm{v}(t)} \mapsto \circledast(t)(\mathcal{P}_h^b)_{(h,b)\in\mathrm{v}(t)}.$$

Namely, for a tree $t : [n] \to \mathcal{O}_{\mathsf{sk}}$ with $[n] = \{0, 1, 2, \dots, n\}$,

$$\circledast(t)(\mathcal{P}_h^b)_{(h,b)\in\mathrm{v}(t)}(z) = \bigoplus_{\forall a\in t(0)\, |\tau(0,a)|=z^a}^{t-\text{tree } \tau} \bigotimes^{h\in I} \bigotimes^{b\in t(h)} \bigotimes^{p\in\tau(h,b)} \mathcal{P}_h^b\Big(\big(|\tau^{-1}_{(h-1,a)\to(h,b)}(p)|\big)_{a\in t_h^{-1}b}\Big).$$

Let $[I] = [n]$, $[J] = [m]$. Correspondingly we denote $I = [I] - \{0\} = \mathbf{n}$, $J = [J] - \{0\} = \mathbf{m}$. Let $f : I \to J$ be an isotonic map. Let isotonic map $\psi = [f] : [J] \to [I]$ viewed as a functor be right adjoint to the functor $[f]^* = (0 \mapsto 0) \sqcup f : [I] \to [J]$. This means that for any $x \in [I]$, $y \in [J]$ the following inequalities are equivalent:

$$x \leqslant [f](y) \Longleftrightarrow [f]^*(x) \leqslant y. \tag{0.12}$$

Formula reads

$$[f] : [J] \to [I], \qquad y \mapsto [f](y) \overset{\text{def}}{=} \max([f]^*)^{-1}([0, y]).$$

Here $[0, y] = \{z \in [J] \mid z \leqslant y\} \subset [J]$. If $t : [I] \to \mathcal{S}_{\mathsf{sk}}$ is a (plain, symmetric or braided) tree, then the composite functor $t_\psi = \big([J] \xrightarrow{\psi} [I] \xrightarrow{t} \mathcal{S}_{\mathsf{sk}}\big)$ is also a tree. If $a, b \in [I]$, $a \leqslant b$, $c \in t(b)$, then the tree $t|^c_{[a,b]} : [a, b] \to \mathcal{S}_{\mathsf{sk}}$ is the subtree of t consisting of vertices in the preimage of c, whose level k is above a: $t|^c_{[a,b]}(k) = t^{-1}_{k\to b}(c)$ for $k \in [a, b]$.

For f and $\psi = [f]$ as above a natural bijection is constructed:

$$\lambda^f : \circledast(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)} \to \circledast(t_\psi)\big(\circledast(t^{|c}_{[\psi(g-1),\psi(g)]})(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t^{|c}_{[\psi(g-1),\psi(g)]})}\big)_{(g,c)\in \mathrm{v}(t_\psi)}.$$

In Section 2 we explain that morphisms with n entries of algebras over operads form an $n \wedge 1$-operad module. In particular, we find this module for A_∞-algebras.

0.28. $n\wedge 1$**-operad modules.** An $n\wedge1$-operad module is a sequence $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$, where $\mathcal{B}$, $\mathcal{A}_i$ are operads for $i \in \mathbf{n}$, and $\mathcal{P} \in \operatorname{Ob} \mathbf{dg}^{\mathbb{N}^n}$ is equipped with unital, associative and commuting actions

$$\rho = \rho_{(k_r)} : \circledast(\mathbf{n} \to \mathbf{1} \to \mathbf{1})(\mathcal{P};\mathcal{B})(\tau_{(k_r)}) = \Big(\bigotimes_{r=1}^{m} \mathcal{P}(k_r)\Big) \otimes \mathcal{B}(m) \to \mathcal{P}\Big(\sum_{r=1}^{m} k_r\Big),$$

$$\tau_\rho = \tau_{(k_r)} = \begin{tikzcd}[row sep=tiny] \mathbf{k}_1^n + \dots + \mathbf{k}_m^n \arrow[dddr] & & \\ \dots \arrow[ddr] & & \\ & & \\ \mathbf{k}_1^2 + \dots + \mathbf{k}_m^2 \arrow[r] & \mathbf{m} \arrow[r] & \mathbf{1} \\ \mathbf{k}_1^1 + \dots + \mathbf{k}_m^1 \arrow[ur] & & \end{tikzcd} \quad , \tag{0.13}$$

$$\lambda = \lambda_{k,(j_p^i)} : \circledast(\mathbf{n} \xrightarrow{1} \mathbf{n} \to \mathbf{1})((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{P})(\tau_{k,(j_p^i)})$$
$$= \Big[\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p^i)\Big] \otimes \mathcal{P}\big((k^i)_{i=1}^n\big) \to \mathcal{P}\Big(\Big(\sum_{p=1}^{k^i} j_p^i\Big)_{i=1}^{n}\Big),$$

$$\tau_\lambda = \tau_{k,(j_p^i)} = \begin{tikzcd}[row sep=small] \sum_{p=1}^{k^n} \mathbf{j}_p^n \arrow[r] & \mathbf{k}^n \arrow[ddr] & \\ \dots \arrow[r] & \dots \arrow[dr] & \\ & & \mathbf{1} \\ \sum_{p=1}^{k^2} \mathbf{j}_p^2 \arrow[r] & \mathbf{k}^2 \arrow[ur] & \\ \sum_{p=1}^{k^1} \mathbf{j}_p^1 \arrow[r] & \mathbf{k}^1 \arrow[uur] & \end{tikzcd} \quad . \tag{0.14}$$

Commutativity and associativity of the above actions can be incorporated into a single requirement: existence. unitality and associativity of the actions

$$\alpha : \circledast(\mathbf{n} \xrightarrow{1} \mathbf{n} \to \mathbf{1} \to \mathbf{1})((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{P};\mathcal{B})(\tau)$$
$$= \Big(\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k_1^i+\dots+k_m^i} \mathcal{A}_i(j_p^i)\Big) \otimes \Big(\bigotimes_{r=1}^{m} \mathcal{P}(k_r)\Big) \otimes \mathcal{B}(m) \to \mathcal{P}\Big(\Big(\sum_{p=1}^{k_1^i+\dots+k_m^i} j_p^i\Big)_{i=1}^{n}\Big),$$

$$\tau_\alpha = \begin{tikzcd}[row sep=small] \sum_{p=1}^{k_1^n+\dots+k_m^n} \mathbf{j}_p^n \arrow[r] & \mathbf{k}_1^n + \dots + \mathbf{k}_m^n \arrow[ddr] & & \\ \dots \arrow[r] & \dots \arrow[dr] & & \\ & & \mathbf{m} \arrow[r] & \mathbf{1} \\ \sum_{p=1}^{k_1^2+\dots+k_m^2} \mathbf{j}_p^2 \arrow[r] & \mathbf{k}_1^2 + \dots + \mathbf{k}_m^2 \arrow[ur] & & \\ \sum_{p=1}^{k_1^1+\dots+k_m^1} \mathbf{j}_p^1 \arrow[r] & \mathbf{k}_1^1 + \dots + \mathbf{k}_m^1 \arrow[uur] & & \end{tikzcd} \tag{0.15}$$

for each $m \in \mathbb{N}$, each family $k_1, \dots, k_m \in \mathbb{N}^n$ and each family of non-negative integers $\bigl((j^i_p)_{p=1}^{k^i_1+\dots+k^i_m}\bigr)_{i=1}^n$. Associativity of α is formulated via contraction of trees.

An example of an $n \wedge 1$-operad module is given by

$$(\mathcal{E}nd\, A_1, \dots, \mathcal{E}nd\, A_n; \mathcal{H}om(A_1, \dots, A_n; B); \mathcal{E}nd\, B),$$

where B, A_i, $i \in \mathbf{n}$, are complexes of $\Bbbk$-modules. The object $\mathcal{H}om((A_i)_{i\in I}; B)$ of $\mathbf{dg}^{\mathbb{N}^I}$, $I = \mathbf{m} \in \mathcal{O}_{\mathsf{sk}}$, is specified by

$$\mathcal{H}om((A_i)_{i\in I}; B)((n^i)_{i\in I}) = \underline{\mathsf{C}}_\Bbbk\bigl(({}^{n^i}A_i)_{i\in I}; B\bigr),$$

where $\underline{\mathsf{C}}_\Bbbk$ is the symmetric $\mathbf{dg}$-multicategory associated with the symmetric closed monoidal category $\mathbf{dg}$. The actions ρ, λ are particular cases of the composition for $\mathcal{H}om$'s, which is defined as the multiplication $\mu_{\underline{\mathsf{C}}_\Bbbk}$ in the symmetric $\mathbf{dg}$-multicategory $\underline{\mathsf{C}}_\Bbbk$. Recall the latter. For a labelled symmetric tree

$$T = \bigl(J \xrightarrow{\phi} P \xrightarrow{\triangleright} \mathbf{1}; (X_j)_{j\in J}, (Y_p)_{p\in P}, Z \mid X_j, Y_p, Z \in \operatorname{Ob}\mathbf{dg}\bigr),$$

$J, P \in \mathcal{O}_{\mathsf{sk}}$, $\phi \in \mathcal{S}\mathrm{et}$, the composition map

$$\mu^T_{\underline{\mathsf{C}}_\Bbbk} : \Bigl(\otimes^{p\in P}\underline{\mathsf{C}}_\Bbbk\bigl((X_j)_{j\in\phi^{-1}p}; Y_p\bigr)\Bigr) \otimes \underline{\mathsf{C}}_\Bbbk\bigl((Y_p)_{p\in P}; Z\bigr) \to \underline{\mathsf{C}}_\Bbbk\bigl((X_j)_{j\in J}; Z\bigr)$$

takes into account the Koszul rule (see Section 0.1) and the symmetry in $\underline{\mathbf{dg}}$. The composition for $\mathcal{H}om$'s given for a tree $t = \bigl(A \xrightarrow{\theta} B \xrightarrow{\triangleright} \mathbf{1}\bigr)$, $\theta \in \mathcal{O}_{\mathsf{sk}}$, labelled with $A \to \operatorname{Ob}\mathbf{dg}$, $a \mapsto X_a$, $B \to \operatorname{Ob}\mathbf{dg}$, $b \mapsto Y_b$, $Z \in \operatorname{Ob}\mathbf{dg}$ and a t-tree $\tau : t \to \mathcal{O}_{\mathsf{sk}}$, $a \mapsto J_a$, $b \mapsto P_b$, $a \mapsto (\tau_a : J_a \to P_{\theta(a)})$ is

$$\begin{aligned}
\mathrm{comp}_\tau :\ & \Bigl[\bigotimes^{b\in B}\bigotimes^{p\in P_b}\mathcal{H}om\bigl((X^a)_{a\in\theta^{-1}b}; Y^b\bigr)\bigl((|\tau_a^{-1}(p)|)_{a\in\theta^{-1}b}\bigr)\Bigr] \otimes \mathcal{H}om\bigl((Y^b)_{b\in B}; Z\bigr)\bigl((|P_b|)_{b\in B}\bigr) \\
= & \Bigl[\bigotimes^{b\in B}\bigotimes^{p\in P_b}\underline{\mathsf{C}}_\Bbbk\bigl(({}^{|\tau_a^{-1}(p)|}X^a)_{a\in\theta^{-1}b}; Y^b\bigr)\Bigr] \otimes \underline{\mathsf{C}}_\Bbbk\bigl(({}^{|P_b|}Y^b)_{b\in B}; Z\bigr) \\
& \qquad \to \underline{\mathsf{C}}_\Bbbk\bigl(({}^{|J_a|}X^a)_{a\in A}; Z\bigr) = \mathcal{H}om\bigl((X^a)_{a\in A}; Z\bigr)\bigl((|J_a|)_{a\in A}\bigr).
\end{aligned}$$

It is defined as the multiplication $\mu^{\tilde{\tau}}_{\underline{\mathsf{C}}_\Bbbk}$, where the labelled symmetric tree

$$\tilde{\tau} = \Bigl(\bigsqcup_{a\in A} J_a \xrightarrow{\tilde{\phi}} \bigsqcup_{b\in B} P_b \xrightarrow{\triangleright} \mathbf{1}\Bigr)$$

is formed by $\tilde{\phi}|_{J_a} = \bigl(J_a \xrightarrow{\tau_a} P_{\theta(a)} \hookrightarrow \sqcup_{b\in B} P_b\bigr)$. The label associated to any $j \in J_a$ (resp. $p \in P_b$, $1 \in \mathbf{1}$) is X^a (resp. Y^b, Z). The trees $t_\rho = (\mathbf{n} \to \mathbf{1} \to \mathbf{1})$, τ_ρ from (0.13) determine $\rho = \mu^{\tilde{\tau}_\rho}_{\underline{\mathsf{C}}_\Bbbk}$ and the trees $t_\lambda = (\mathbf{n} \xrightarrow{1} \mathbf{n} \to \mathbf{1})$, τ_λ from (0.14) determine $\lambda = \mu^{\tilde{\tau}_\lambda}_{\underline{\mathsf{C}}_\Bbbk}$.

Given an operad $\mathcal{O}$ and an $n \wedge 1$-operad $\mathcal{O}$-module $\mathcal{F}_n$ for each $n \geqslant 0$ we define a *morphism of* $\mathcal{O}$*-algebras with* n *arguments* $X_1, \dots, X_n \to Y$ as a morphism of ${}_n\mathrm{Op}_1$

$$(\mathcal{O}, \dots, \mathcal{O}; \mathcal{F}_n; \mathcal{O}) \to (\mathcal{E}nd\, X_1, \dots, \mathcal{E}nd\, X_n; \mathcal{H}om(X_1, \dots, X_n; Y); \mathcal{E}nd\, Y).$$

0.29. Resolution $F_n \to FAs_n$. Consider a $n \wedge 1$-operad As-module FAs_n having $FAs_n(j^1,\dots,j^n) = \Bbbk$ for all non–vanishing $(j^1,\dots,j^n) \in \mathbb{N}^n$, while $FAs_n(0,\dots,0) = 0$. The actions for FAs_n are given by multiplication in $\Bbbk$. A morphism of $n \wedge 1$-operad modules

$$(As,\dots,As;FAs_n;As) \to (\mathcal{E}nd\,A_1,\dots,\mathcal{E}nd\,A_n;\mathcal{H}om(A_1,\dots,A_n;B);\mathcal{E}nd\,B),$$
$$\mathcal{H}om(A_1,\dots,A_n;B)(j^1,\dots,j^n) = \underline{\mathbf{dg}}(A_1^{\otimes j^1}\otimes\dots\otimes A_n^{\otimes j^n},B),$$

amounts to a family of morphisms $f_i : A_i \to B$ of associative differential graded $\Bbbk$-algebras without units, $i \in \mathbf{n}$, such that the following diagrams commute for all $1 \leqslant i < j \leqslant n$:

$$\begin{tikzcd}
A_i\otimes A_j \arrow[r, "c", "\sim"'] \arrow[d, "f_i\otimes f_j"'] & A_j\otimes A_i \arrow[r, "f_j\otimes f_i"] & B\otimes B \arrow[d, "m_B"] \\
B\otimes B \arrow[rr, "m_B"] & & B
\end{tikzcd}$$

see Example 2.3.

0.30 Theorem (Propositions 2.7, 2.10 and Theorem 2.14)**.** *The $n \wedge 1$-operad module (As, FAs_n) admits a cofibrant replacement $(\mathrm{A}_\infty, \mathrm{F}_n)$, where the $n \wedge 1$-operad A_∞-module $\mathrm{F}_n = \boxdot_{\geqslant 0}({}^n\mathrm{A}_\infty; \Bbbk\{\mathsf{f}_j \mid j \in \mathbb{N}^n - 0\}; \mathrm{A}_\infty)$ is freely generated as a graded module by elements $\mathsf{f}_{j^1,\dots,j^n} \in \mathrm{F}_n(j^1,\dots,j^n)$, $(j^1,\dots,j^n) \in \mathbb{N}^n - 0$, of degree $1 - j^1 - \dots - j^n = 1 - |j|$. When these generators are taken into $\mathcal{H}om(A_1,\dots,A_n;B)$ they become linear maps $\boxtimes^{i\in\mathbf{n}} T^{\ell^i}A_i \to B$. The definition of the differential is*

$$\mathsf{f}_\ell\partial = \sum_{q=1}^{n}\sum_{r+x+t=\ell^q}^{x>1} (-1)^{(1-x)(\ell^1+\dots+\ell^{q-1}+r)+1-|\ell|}\lambda^q_{({}^r1,x,{}^t1)}({}^r1,m_x,{}^t1;\mathsf{f}_{\ell-(x-1)e_q})$$
$$+\sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>1} (-1)^{k+\sum_{1\leqslant b<a\leqslant k}^{1\leqslant c<d\leqslant n} j_a^c j_b^d+\sum_{p=1}^k(p-1)(|j_p|-1)}\rho_{(j_p^i)}((\mathsf{f}_{j_p})_{p=1}^k;m_k).$$

The first arguments of λ are all $1 = \mathrm{id}$ except m_x on the only place $p = r+1$. Moreover, $(\mathrm{A}_\infty,\mathrm{F}_n) \to (As, FAs_n)$ is a homotopy isomorphism.

The differential interpreted on A_∞-algebras A_1, …, A_n, B means

$$\mathsf{f}_\ell\partial = \sum_{q=1}^{n}\sum_{r+x+t=\ell^q}^{x>1} (-1)^{(1-x)(\ell^1+\dots+\ell^{q-1}+r)+1-|\ell|}\big[\otimes^{i\in\mathbf{n}}T^{\ell^i}A_i \xrightarrow{1^{\otimes(q-1)}\otimes(1^{\otimes r}\otimes m_x\otimes 1^{\otimes t})\otimes 1^{\otimes(n-q)}}$$
$$T^{\ell^1}A_1\otimes\dots\otimes T^{\ell^{q-1}}A_{q-1}\otimes T^{r+1+t}A_q\otimes T^{\ell^{q+1}}A_{q+1}\otimes\dots\otimes T^{\ell^n}A_n \xrightarrow{\mathsf{f}_{\ell-(x-1)e_q}} B\big]$$
$$+\sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>1} (-1)^{k+\sum_{1\leqslant b<a\leqslant k}^{1\leqslant c<d\leqslant n} j_a^c j_b^d+\sum_{p=1}^k(p-1)(|j_p|-1)}\big[\otimes^{i\in\mathbf{n}}T^{\ell^i}A_i \xrightarrow{\sim} \otimes^{i\in\mathbf{n}}\otimes^{p\in\mathbf{k}}T^{j_p^i}A_i \xrightarrow{\sim}$$
$$\otimes^{p\in\mathbf{k}}\otimes^{i\in\mathbf{n}}T^{j_p^i}A_i \xrightarrow{\otimes^{p\in\mathbf{k}}\mathsf{f}_{j_p}} \otimes^{p\in\mathbf{k}}B \xrightarrow{m_k} B\big].$$

0.31. Morphisms of A_∞-algebras with n arguments. F_n-algebra maps consist of A_∞-algebras $A_1, \dots, A_n$, B, and an A_∞-morphism $(\mathsf{f}_j)_{j\in\mathbb{N}^n-0}$. The latter means that the equation holds for all $\ell \in \mathbb{N}^n - 0$:

$$\mathsf{f}_\ell m_1 + (-1)^{|\ell|}\Bigl[\sum_{q=1}^{n}\sum_{r+1+t=\ell^q} 1^{\otimes(q-1)} \otimes (1^{\otimes r}\otimes m_1 \otimes 1^{\otimes t})\otimes 1^{\otimes(n-q)}\Bigr]\mathsf{f}_\ell = \mathsf{f}_\ell\partial.$$

This is equivalent to equation (0.10).

Composition of A_∞-morphisms can be obtained from comultiplication in the system F_n. We view it as a coalgebra and $\mathcal{H}om$ as an algebra. Then homomorphisms between them form an algebra as well. This way multiquiver a_∞ whose objects are A_∞-algebras $(B, \alpha_B : \mathrm{A}_\infty \to \mathcal{E}nd\, B)$ with the set of morphisms

$$\begin{aligned}&\mathsf{a}_\infty((A_i,\alpha_{A_i})_{i\in I};(B,\alpha_B))\\ &\quad = \{((\alpha_{A_i})_{i\in I};\phi;\alpha_B) : ({}^I\mathrm{A}_\infty;\mathrm{F}_I; mainf) \to ((\mathcal{E}nd\,A_i)_{i\in I};\mathcal{H}om((A_i)_{i\in I};B);\mathcal{E}nd\,B)\}\end{aligned}$$

becomes a multicategory.

For any tree t and any collection of A_∞-algebras $\alpha_h^b : \mathrm{A}_\infty \to \mathcal{E}nd\, A_h^b$, $(h,b) \in \overline{\mathrm{v}}(t)$, assume given $t_h^{-1}b \wedge 1$-operad module morphisms for $(h,b) \in \mathrm{v}(t)$:

$$\begin{aligned}&\bigl((\alpha^a_{h-1})_{a\in t_h^{-1}b};g_h^b;\alpha_h^b\bigr) : \bigl({}^{t_h^{-1}b}\mathrm{A}_\infty;\mathrm{F}_{t_h^{-1}b};\mathrm{A}_\infty\bigr)\\ &\qquad\qquad \to \bigl((\mathcal{E}nd\,A^a_{h-1})_{a\in t_h^{-1}b};\mathcal{H}om((A^a_{h-1})_{a\in t_h^{-1}b};A_h^b);\mathcal{E}nd\,A_h^b\bigr).\end{aligned}$$

Their composition can be defined as $\bigl((\alpha_0^a)_{a\in t(0)};\mathrm{comp}(t)(g_h^b);\alpha_l^1\bigr)$, where

$$\begin{aligned}\mathrm{comp}(t)(g_h^b) = \bigl[&\mathrm{F}_{t(0)} \xrightarrow{\Delta(t)} \circledast(t)(\mathrm{F}_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)} \xrightarrow{\circledast(t)(g_h^b)}\\ &\circledast(t)\bigl(\mathcal{H}om((A^a_{h-1})_{a\in t_h^{-1}b};A_h^b)\bigr)_{(h,b)\in\mathrm{v}(t)} \xrightarrow{\mathrm{comp}(t)} \mathcal{H}om((A_0^a)_{a\in t(0)};A_l^1)\bigr].\end{aligned}$$

Explicit form of the comultiplication is

$$\Delta(t)(\mathsf{f}_j) = \sum_{\forall a\in t(0)\,|\tau(0,a)|=j^a}^{t-\text{tree }\tau} (-1)^{c(\tilde\tau)} \otimes^{h\in I}\otimes^{b\in t(h)}\otimes^{p\in\tau(h,b)} \mathsf{f}_{|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}}.$$

The sign $c(\tilde\tau)$ is computed via recipe (1.21) through the Koszul rule.

1. Lax $\mathcal{C}at$-span multicategories

In this section we describe the categorical background to the main subject of morphisms with several entries.

1.1. $\mathcal{C}at$-spans. A *$\mathcal{C}at$-span* C is a pair of functors with a common source, which we denote ${}^{\mathsf{s}}\mathsf{C} \xleftarrow{\mathrm{src}} \mathsf{C} \xrightarrow{\mathrm{tgt}} {}_{\mathsf{t}}\mathsf{C}$. We say that ${}^{\mathsf{s}}\mathsf{C}$ is the source category and ${}_{\mathsf{t}}\mathsf{C}$ is the target category. For example, anafunctors (see Makkai [Mak96]) are particular instances of $\mathcal{C}at$-spans. A *morphism of $\mathcal{C}at$-spans* $F : \mathsf{A} \to \mathsf{B}$ is a triple of functors

$$
{}^{\mathsf{s}}F : {}^{\mathsf{s}}\mathsf{A} \to {}^{\mathsf{s}}\mathsf{B}, \qquad F : \mathsf{A} \to \mathsf{B}, \qquad {}_{\mathsf{t}}F : {}_{\mathsf{t}}\mathsf{A} \to {}_{\mathsf{t}}\mathsf{B},
$$

strictly commuting with the source functor src and the target functor tgt:

$$
\begin{tikzcd}
{}^{\mathsf{s}}\mathsf{A} \arrow[d, "{}^{\mathsf{s}}F"'] \arrow[r, phantom, ""] & \mathsf{A} \arrow[l, "\mathrm{src}"'] \arrow[r, "\mathrm{tgt}"] \arrow[d, "F"'] & {}_{\mathsf{t}}\mathsf{A} \arrow[d, "{}_{\mathsf{t}}F"] \\
{}^{\mathsf{s}}\mathsf{B} & \mathsf{B} \arrow[l, "\mathrm{src}"'] \arrow[r, "\mathrm{tgt}"] & {}_{\mathsf{t}}\mathsf{B}
\arrow[from=1-1, to=2-2, phantom, "="] \arrow[from=1-2, to=2-3, phantom, "="]
\end{tikzcd}
\tag{1.1}
$$

So described category $\mathcal{C}at$-span has arbitrary products. In fact,

$$
\prod_{i\in I}({}^{\mathsf{s}}\mathsf{C}_i \xleftarrow{\mathrm{src}} \mathsf{C}_i \xrightarrow{\mathrm{tgt}} {}_{\mathsf{t}}\mathsf{C}_i) = \Big(\prod_{i\in I}{}^{\mathsf{s}}\mathsf{C}_i \xleftarrow{\prod \mathrm{src}} \prod_{i\in I}\mathsf{C}_i \xrightarrow{\prod \mathrm{tgt}} \prod_{i\in I}{}_{\mathsf{t}}\mathsf{C}_i\Big).
$$

First proof of the following statement was given by Sergiy Slobodianiuk (unpublished). The proof presented here is devised by the author.

1.2 Proposition. *The category* $\mathcal{C}at$-span *is Cartesian closed.*

Proof. The inner homomorphisms object $\underline{\mathcal{C}at\text{-span}}(\mathsf{A},\mathsf{B})$ for $(\mathcal{C}at\text{-span},\times)$ is given by the pair of functors $\underline{\mathcal{C}at}({}^{\mathsf{s}}\mathsf{A},{}^{\mathsf{s}}\mathsf{B}) \xleftarrow[\mathrm{pr}_1]{\mathrm{src}} \underline{cat\text{-span}}(\mathsf{A},\mathsf{B}) \xrightarrow[\mathrm{pr}_3]{\mathrm{tgt}} \underline{\mathcal{C}at}({}_{\mathsf{t}}\mathsf{A},{}_{\mathsf{t}}\mathsf{B})$, where objects of the category $\underline{cat\text{-span}}(\mathsf{A},\mathsf{B})$ are triples of functors $({}^{\mathsf{s}}F, F, {}_{\mathsf{t}}F)$ such that diagram (1.1) commutes. Morphisms $({}^{\mathsf{s}}F, F, {}_{\mathsf{t}}F) \to ({}^{\mathsf{s}}G, G, {}_{\mathsf{t}}G)$ of the category $\underline{cat\text{-span}}(\mathsf{A},\mathsf{B})$ are triples of natural transformations

$$
\big(({}^{\mathsf{s}}\phi : {}^{\mathsf{s}}F \to {}^{\mathsf{s}}G : {}^{\mathsf{s}}\mathsf{A} \to {}^{\mathsf{s}}\mathsf{B}), (\phi : F \to G : \mathsf{A} \to \mathsf{B}), ({}_{\mathsf{t}}\phi : {}_{\mathsf{t}}F \to {}_{\mathsf{t}}G : {}_{\mathsf{t}}\mathsf{A} \to {}_{\mathsf{t}}\mathsf{B})\big)
$$

such that

$$
\begin{aligned}
&\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathrm{src}"] \arrow[d, shift right=3, "F"'] \arrow[d, shift left=3, "G"] \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & {}^{\mathsf{s}}\mathsf{A} \arrow[d, "{}^{\mathsf{s}}G"] \\
\mathsf{B} \arrow[r, "\mathrm{src}"] & {}^{\mathsf{s}}\mathsf{B}
\arrow[from=1-1, to=2-2, phantom, "="]
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathrm{src}"] \arrow[d, "F"'] & {}^{\mathsf{s}}\mathsf{A} \arrow[d, shift right=3, "{}^{\mathsf{s}}F"'] \arrow[d, shift left=3, "{}^{\mathsf{s}}G"] \arrow[d, phantom, "\overset{{}^{\mathsf{s}}\phi}{\Rightarrow}"] \\
\mathsf{B} \arrow[r, "\mathrm{src}"] & {}^{\mathsf{s}}\mathsf{B}
\arrow[from=1-1, to=2-2, phantom, "="]
\end{tikzcd}
, \\
&\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathrm{tgt}"] \arrow[d, shift right=3, "F"'] \arrow[d, shift left=3, "G"] \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & {}_{\mathsf{t}}\mathsf{A} \arrow[d, "{}_{\mathsf{t}}G"] \\
\mathsf{B} \arrow[r, "\mathrm{tgt}"] & {}_{\mathsf{t}}\mathsf{B}
\arrow[from=1-1, to=2-2, phantom, "="]
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{A} \arrow[r, "\mathrm{tgt}"] \arrow[d, "F"'] & {}_{\mathsf{t}}\mathsf{A} \arrow[d, shift right=3, "{}_{\mathsf{t}}F"'] \arrow[d, shift left=3, "{}_{\mathsf{t}}G"] \arrow[d, phantom, "\overset{{}_{\mathsf{t}}\phi}{\Rightarrow}"] \\
\mathsf{B} \arrow[r, "\mathrm{tgt}"] & {}_{\mathsf{t}}\mathsf{B}
\arrow[from=1-1, to=2-2, phantom, "="]
\end{tikzcd}
.
\end{aligned}
\tag{1.2}
$$

The source and target functors are projections $\mathrm{src} = \mathrm{pr}_1 : ({}^{\mathsf{s}}F, F, {}_{\mathsf{t}}F) \mapsto {}^{\mathsf{s}}F$, $({}^{\mathsf{s}}\phi, \phi, {}_{\mathsf{t}}\phi) \mapsto {}^{\mathsf{s}}\phi$, and $\mathrm{tgt} = \mathrm{pr}_3 : ({}^{\mathsf{s}}F, F, {}_{\mathsf{t}}F) \mapsto {}_{\mathsf{t}}F$, $({}^{\mathsf{s}}\phi, \phi, {}_{\mathsf{t}}\phi) \mapsto {}_{\mathsf{t}}\phi$. The evaluation morphism $\mathrm{ev} : \mathsf{A} \times \underline{\mathcal{C}at\text{-span}}(\mathsf{A}, \mathsf{B}) \to \mathsf{B}$ consists of three functors

$$\bigl({}^{\mathsf{s}}\mathsf{A} \times \underline{\mathcal{C}at}({}^{\mathsf{s}}\mathsf{A}, {}^{\mathsf{s}}\mathsf{B}) \xrightarrow{\mathrm{ev}} {}^{\mathsf{s}}\mathsf{B}, \mathsf{A} \times \mathsf{D} \xrightarrow{1\times\mathrm{pr}_2} \mathsf{A} \times \underline{\mathcal{C}at}(\mathsf{A}, \mathsf{B}) \xrightarrow{\mathrm{ev}} \mathsf{B}, {}_{\mathsf{t}}\mathsf{A} \times \underline{\mathcal{C}at}({}_{\mathsf{t}}\mathsf{A}, {}_{\mathsf{t}}\mathsf{B}) \xrightarrow{\mathrm{ev}} {}_{\mathsf{t}}\mathsf{B}\bigr),$$

where $\mathsf{D} = \underline{cat\text{-span}}(\mathsf{A}, \mathsf{B})$.

Notice that there is a 2-category $\underline{cat\text{-span}}$, whose objects are $\mathcal{C}at$-spans and whose categories of morphisms are $\underline{cat\text{-span}}(\mathsf{A}, \mathsf{B})$. All morphisms in this 2-category, including left and right whiskering, are compositions in $\mathcal{C}at$, performed simultaneously in three places: the source, the main body and the target. Such compositions preserve commutation relations (1.1), (1.2), hence, they give well-defined compositions in $\underline{cat\text{-span}}$. Standard equations involving them hold in $\underline{cat\text{-span}}$, since they hold in the 2-category $\underline{\mathcal{C}at}$.

The underlying category of objects and 1-morphisms in $\underline{cat\text{-span}}$ is precisely $\mathcal{C}at$-span. Furthermore, finite products in $\mathcal{C}at$-span extend to a (symmetric) monoidal 2-category structure of $\underline{cat\text{-span}}$, just as they do for $\mathcal{C}at$ and $\underline{\mathcal{C}at}$.

We have to prove that the mapping

$$\begin{aligned} \varphi = \bigl[&\mathcal{C}at\text{-span}(\mathsf{C}, \underline{\mathcal{C}at\text{-span}}(\mathsf{A}, \mathsf{B})) \xrightarrow{(1_{\mathsf{A}}\times-)\times\mathrm{ev}} \\ &\mathcal{C}at\text{-span}(\mathsf{A}\times\mathsf{C}, \mathsf{A}\times\underline{\mathcal{C}at\text{-span}}(\mathsf{A}, \mathsf{B})) \times \mathcal{C}at\text{-span}(\mathsf{A}\times\underline{\mathcal{C}at\text{-span}}(\mathsf{A}, \mathsf{B}), \mathsf{B}) \\ &\xrightarrow{\mathrm{comp}} \mathcal{C}at\text{-span}(\mathsf{A}\times\mathsf{C}, \mathsf{B})\bigr] \end{aligned} \tag{1.3}$$

is a bijection. Let us construct a map inverse to φ.

The unit object of $(\mathcal{C}at\text{-span}, \times)$ and $(\underline{cat\text{-span}}, \times)$ is the terminal $\mathcal{C}at$-span $\mathbb{1}^3 = (\mathbb{1} \leftarrow \mathbb{1} \to \mathbb{1})$, where $\mathbb{1}$ is the terminal category. A morphism $\mathbb{1}^3 \to \mathsf{C}$ of $\mathcal{C}at$-span has the form $(* \mapsto \mathrm{src}\, X, * \mapsto X, * \mapsto \mathrm{tgt}\, X)$ for some object $X \in \mathrm{Ob}\,\mathsf{C}$. Denote it by $\dot X : \mathbb{1}^3 \to \mathsf{C}$. A 2-morphism $\dot X \to \dot Y : \mathbb{1}^3 \to \mathsf{C}$ of $\underline{cat\text{-span}}$ identifies with a morphism $f : X \to Y \in \mathsf{C}$. Denote it by $\dot f : \dot X \to \dot Y : \mathbb{1}^3 \to \mathsf{C}$, $(* \mapsto \mathrm{src}\, f, * \mapsto f, * \mapsto \mathrm{tgt}\, f)$.

Given a $\mathcal{C}at$-span morphism $F : \mathsf{A}\times\mathsf{C} \to \mathsf{B}$, let us construct a $\mathcal{C}at$-span morphism $\Psi(F) : \mathsf{C} \to \underline{\mathcal{C}at\text{-span}}(\mathsf{A}, \mathsf{B})$. First of all, for any object X of the category C there is a morphism of $\mathcal{C}at$-spans

$$\mathsf{A} \xrightarrow{\sim} \mathsf{A}\times\mathbb{1}^3 \xrightarrow{1\times\dot X} \mathsf{A}\times\mathsf{C} \xrightarrow{F} \mathsf{B},$$

given by three functors

$$({}^{\mathsf{s}}F(-, \mathrm{src}\, X), F(-, X), {}_{\mathsf{t}}F(-, \mathrm{tgt}\, X)).$$

This is an object $\psi(F)(X)$ of $\underline{cat\text{-span}}(\mathsf{A}, \mathsf{B})$. Secondly, for any morphism $f : X \to Y$ of the category C there is a 2-morphism of $\underline{cat\text{-span}}$

$$\Bigl(\mathsf{A} \xrightarrow{\sim} \mathsf{A}\times\mathbb{1}^3 \underset{1\times\dot Y}{\overset{1\times\dot X}{\underset{1\times\dot f\Downarrow}{\rightrightarrows}}} \mathsf{A}\times\mathsf{C} \xrightarrow{F} \mathsf{B}\Bigr) = \Bigl(\psi(F)(f) : \psi(F)(X) \to \psi(F)(Y)\Bigr)$$

given by three natural transformations

$$({}^{\mathsf{s}}F(-,\operatorname{src} f), F(-,f), {}_{\mathsf{t}}F(-,\operatorname{tgt} f)).$$

Clearly, this determines a functor

$$\psi(F) : \mathsf{C} \to \underline{cat\text{-span}}(\mathsf{A},\mathsf{B}).$$

On the other hand, given functors ${}^{\mathsf{s}}F : {}^{\mathsf{s}}\mathsf{A} \times {}^{\mathsf{s}}\mathsf{C} \to {}^{\mathsf{s}}\mathsf{B}$ and ${}_{\mathsf{t}}F : {}_{\mathsf{t}}\mathsf{A} \times {}_{\mathsf{t}}\mathsf{C} \to {}_{\mathsf{t}}\mathsf{B}$ induce by closedness of $(\mathcal{C}at, \times)$ the functors

$$\begin{aligned} \overline{{}^{\mathsf{s}}F} : {}^{\mathsf{s}}\mathsf{C} &\longrightarrow \underline{\mathcal{C}at}({}^{\mathsf{s}}\mathsf{A},{}^{\mathsf{s}}\mathsf{B}) \\ U &\longmapsto {}^{\mathsf{s}}F(-,U) \end{aligned}, \qquad \begin{aligned} \overline{{}_{\mathsf{t}}F} : {}_{\mathsf{t}}\mathsf{C} &\longrightarrow \underline{\mathcal{C}at}({}_{\mathsf{t}}\mathsf{A},{}_{\mathsf{t}}\mathsf{B}) \\ V &\longmapsto {}_{\mathsf{t}}F(-,V) \end{aligned}.$$

Explicit expressions show that both squares of the diagram

$$\begin{tikzcd} {}^{\mathsf{s}}\mathsf{C} \arrow[d, "\overline{{}^{\mathsf{s}}F}"'] & \mathsf{C} \arrow[l, "\mathrm{src}"'] \arrow[r, "\mathrm{tgt}"] \arrow[d, "\psi(F)"] & {}_{\mathsf{t}}\mathsf{C} \arrow[d, "\overline{{}_{\mathsf{t}}F}"] \\ \underline{\mathcal{C}at}({}^{\mathsf{s}}\mathsf{A},{}^{\mathsf{s}}\mathsf{B}) & \underline{cat\text{-span}}(\mathsf{A},\mathsf{B}) \arrow[l, "\mathrm{src}", "\mathrm{pr}_1"'] \arrow[r, "\mathrm{tgt}", "\mathrm{pr}_3"'] & \underline{\mathcal{C}at}({}_{\mathsf{t}}\mathsf{A},{}_{\mathsf{t}}\mathsf{B}) \end{tikzcd}$$

commute. Therefore, there is a $\mathcal{C}at$-span morphism

$$\Psi(F) = (\overline{{}^{\mathsf{s}}F}, \psi(F), \overline{{}_{\mathsf{t}}F}) : \mathsf{C} \to \underline{\mathcal{C}at\text{-span}}(\mathsf{A},\mathsf{B}).$$

This gives a map

$$\Psi : \mathcal{C}at\text{-span}(\mathsf{A}\times\mathsf{C},\mathsf{B}) \to \mathcal{C}at\text{-span}(\mathsf{C}, \underline{\mathcal{C}at\text{-span}}(\mathsf{A},\mathsf{B})).$$

Without much efforts one checks that it is inverse to mapping φ given by (1.3). □

Thus there is a category $\underline{\mathcal{C}at\text{-span}}$ enriched in $\mathcal{C}at$-span. The underlying functor pr_2 : $\mathcal{C}at$-span $\to \mathcal{C}at$, $({}^{\mathsf{s}}\mathsf{C} \xleftarrow{\mathrm{src}} \mathsf{C} \xrightarrow{\mathrm{tgt}} {}_{\mathsf{t}}\mathsf{C}) \mapsto \mathsf{C}$ turns $\mathcal{C}at$-span into a $\mathcal{C}at$-category. This 2-category structure of $\mathcal{C}at$-span coincides with that of $\underline{cat\text{-span}}$ discussed in the above proof. In particular, a 2-morphism of $\mathcal{C}at$-span is a triple $\overline{({}^{\mathsf{s}}\phi, \phi, {}_{\mathsf{t}}\phi)}$ which satisfies (1.2).

1.3. $\mathcal{C}at$-span multiquivers. Consider the category StrictMon$\mathcal{C}at$ of strict monoidal categories and strict monoidal functors. The underlying functor U and the 'free strict monoidal category' functor F form an adjunction $F : \mathcal{C}at \rightleftarrows \mathrm{StrictMon}\mathcal{C}at : U$. The monad -$^* = U \circ F : \mathcal{C}at \to \mathcal{C}at$ associated with this adjunction takes a category $\mathcal{C}$ to $\mathcal{C}^* = \sqcup_{n\in\mathbb{N}} \mathcal{C}^n$. This 'free strict monoidal category' monad is cartesian, see Definition 4.1.1 and Example 4.1.15 of [Lei03]. Thus, the monad -* is suitable for introducing a kind of multicategories.

A *$\mathcal{C}at$-span multiquiver* is a $\mathcal{C}at$-span C together with ${}_{\mathsf{s}}\mathsf{C}$ such that ${}^{\mathsf{s}}\mathsf{C} = ({}_{\mathsf{s}}\mathsf{C})^* = \sqcup_{n\in\mathbb{N}}\, {}_{\mathsf{s}}\mathsf{C}^n$, thus,

$$
{}^{\mathsf{s}}\mathsf{C}((X_i)_{i\in\mathbf{n}}, (Y_j)_{j\in\mathbf{m}}) = \begin{cases} \prod_{i\in\mathbf{n}} {}_{\mathsf{s}}\mathsf{C}(X_i, Y_i), & \text{if } n = m, \\ \varnothing, & \text{otherwise.} \end{cases}
$$

Since ${}^{\mathsf{s}}\mathsf{C}$ is a disjoint union of categories ${}_{\mathsf{s}}\mathsf{C}^n$, the mere existence of a functor $\mathrm{src} : \mathsf{C} \to {}^{\mathsf{s}}\mathsf{C}$ implies that $\mathsf{C} = \sqcup_{n\in\mathbb{N}} \mathsf{C}(n)$ is a disjoint union as well, where $\mathsf{C}(n) = \mathrm{src}^{-1}({}_{\mathsf{s}}\mathsf{C}^n)$. Thus we can view a $\mathcal{C}at$-span multiquiver as a sequence of pairs of functors

$$
\bigl({}_{\mathsf{s}}\mathsf{C}^n \xleftarrow{\mathrm{src}} \mathsf{C}(n) \xrightarrow{\mathrm{tgt}} {}_{\mathsf{t}}\mathsf{C} \mid n \geqslant 0\bigr).
$$

A *morphism of $\mathcal{C}at$-span multiquivers* is a morphism of $\mathcal{C}at$-spans $F : \mathsf{C} \to \mathsf{D}$ together with ${}_{\mathsf{s}}F : {}_{\mathsf{s}}\mathsf{C} \to {}_{\mathsf{s}}\mathsf{D}$ such that ${}^{\mathsf{s}}F = ({}_{\mathsf{s}}F)^*$. Necessarily $F = \bigl(F(n) : \mathsf{C}(n) \to \mathsf{D}(n) \mid n \geqslant 0\bigr)$. This defines the category $\mathfrak{SMQ}_{\mathcal{C}at}$ of $\mathcal{C}at$-span multiquivers.

Moreover, $\mathfrak{SMQ}_{\mathcal{C}at}$ is a 2-category and has a strict 2-functor $\mathfrak{SMQ}_{\mathcal{C}at} \to \mathcal{C}at$-span, injective on objects, 1-morphisms and 2-morphisms. Namely, a *2-morphism of $\mathcal{C}at$-span multiquiver morphisms* $\phi : F \to G : \mathsf{C} \to \mathsf{D}$ is a triple of natural transformations ${}_{\mathsf{s}}\phi : {}_{\mathsf{s}}F \to {}_{\mathsf{s}}G : {}_{\mathsf{s}}\mathsf{C} \to {}_{\mathsf{s}}\mathsf{D}$, ϕ and ${}_{\mathsf{t}}\phi : {}_{\mathsf{t}}F \to {}_{\mathsf{t}}G : {}_{\mathsf{t}}\mathsf{C} \to {}_{\mathsf{t}}\mathsf{D}$ such that the triple $(({}_{\mathsf{s}}\phi)^*, \phi, {}_{\mathsf{t}}\phi)$ satisfies (1.2):

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathrm{tgt}"] \arrow[d, shift right=3, "F"'] \arrow[d, shift left=3, "G"] \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & {}_{\mathsf{t}}\mathsf{C} \arrow[d, "{}_{\mathsf{t}}G"] \\
\mathsf{D} \arrow[r, "\mathrm{tgt}"'] & {}_{\mathsf{t}}\mathsf{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathrm{tgt}"] \arrow[d, "F"'] & {}_{\mathsf{t}}\mathsf{C} \arrow[d, shift right=3, "{}_{\mathsf{t}}F"'] \arrow[d, shift left=3, "{}_{\mathsf{t}}G"] \arrow[d, phantom, "\overset{{}_{\mathsf{t}}\phi}{\Rightarrow}"] \\
\mathsf{D} \arrow[r, "\mathrm{tgt}"'] & {}_{\mathsf{t}}\mathsf{D}
\end{tikzcd}
,
\tag{1.4}
$$

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathrm{src}"] \arrow[d, shift right=3, "F"'] \arrow[d, shift left=3, "G"] \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] & ({}_{\mathsf{s}}\mathsf{C})^* \arrow[d, "({}_{\mathsf{s}}G)^*"] \\
\mathsf{D} \arrow[r, "\mathrm{src}"'] & ({}_{\mathsf{s}}\mathsf{D})^*
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathrm{src}"] \arrow[d, "F"'] & ({}_{\mathsf{s}}\mathsf{C})^* \arrow[d, shift right=3, "({}_{\mathsf{s}}F)^*"'] \arrow[d, shift left=3, "({}_{\mathsf{s}}G)^*"] \arrow[d, phantom, "\overset{({}_{\mathsf{s}}\phi)^*}{\Longrightarrow}"] \\
\mathsf{D} \arrow[r, "\mathrm{src}"'] & ({}_{\mathsf{s}}\mathsf{D})^*
\end{tikzcd}
.
$$

1.4. Spans and related structures. Consider the category strict-2-$\mathcal{C}at = \mathcal{C}at$-$\mathcal{C}at$ of strict 2-categories and strict 2-functors. It is complete, so there exists a bicategory $\mathcal{S}\mathrm{pan}\,\mathcal{C}at$-$\mathcal{C}at$ of spans in it. We shall need the category $\mathscr{S} = \mathcal{S}\mathrm{pan}\,\mathcal{C}at$-$\mathcal{C}at(\mathcal{C}at, \mathcal{C}at)$, whose objects are pairs of strict 2-functors $\mathcal{S} = \bigl(\mathcal{C}at \xleftarrow{\mathrm{src}} \mathcal{S} \xrightarrow{\mathrm{tgt}} \mathcal{C}at\bigr)$, and morphisms $F : \mathcal{S} \to \mathcal{P}$ are commutative diagrams of strict 2-functors

$$
\begin{tikzcd}
\mathcal{C}at \arrow[d, equal] \arrow[dr, phantom, "="] & \mathcal{S} \arrow[l, "\mathrm{src}"'] \arrow[r, "\mathrm{tgt}"] \arrow[d, "F"] \arrow[dr, phantom, "="] & \mathcal{C}at \arrow[d, equal] \\
\mathcal{C}at & \mathcal{P} \arrow[l, "\mathrm{src}"] \arrow[r, "\mathrm{tgt}"'] & \mathcal{C}at
\end{tikzcd}
$$

By the general theory of bicategories the category $\mathscr{S}$ is Monoidal, the term is suggested in [BLM08, Definition 2.5]. Its monoidal product is

$$
\boxdot^{h\in\mathbf{n}} \mathcal{S}_h = \lim(\mathcal{C}at \xleftarrow{\mathrm{src}} \mathcal{S}_1 \xrightarrow{\mathrm{tgt}} \mathcal{C}at \xleftarrow{\mathrm{src}} \mathcal{S}_2 \xrightarrow{\mathrm{tgt}} \dots \mathcal{C}at \xleftarrow{\mathrm{src}} \mathcal{S}_n \xrightarrow{\mathrm{tgt}} \mathcal{C}at). \tag{1.5}
$$

In this diagram the limit is taken in strict-2-$\mathcal{C}at$. The canonical strict 2-functors to the first and the last $\mathcal{C}at$-vertices are the horizontal source and target of $\boxbox^{h\in\mathbf{n}}\mathcal{S}_h$. Of course, $\boxbox^{\varnothing} = \big(\mathcal{C}at \xleftarrow{\mathrm{Id}} \mathcal{C}at \xrightarrow{\mathrm{Id}} \mathcal{C}at\big)$. For any non-decreasing map $\phi : \mathbf{n} \to \mathbf{k}$ there are isomorphisms in strict-2-$\mathcal{C}at$ $\boxbox^{h\in\mathbf{n}}\mathcal{S}_h \xrightarrow{\sim} \boxbox^{m\in\mathbf{k}}\,\boxbox^{h\in\phi^{-1}m}\mathcal{S}_h$ due to presenting repeated limits as a single limit. The isomorphism $\boxbox^1\mathcal{S} = \lim(\mathcal{C}at \leftarrow \mathcal{S} \to \mathcal{C}at) = \mathcal{S}$ is chosen to be the identity.

Now we shall equip $\mathcal{SMQ}_{\mathcal{C}at}$ with some structures, which will turn it later into a weak triple $\mathcal{C}at$-category. The latter notion will be defined below as close to the strict one as possible. Such a version is just what we need for the purposes of this article. Thus, $\mathcal{SMQ}_{\mathcal{C}at}$ is equipped with the following data:

1. Horizontal source and target, a pair of strict 2-functors

$$
\begin{array}{ccccc}
\mathcal{C}at & \xleftarrow{\quad\mathrm{pr}_1\quad} & \mathcal{SMQ}_{\mathcal{C}at} & \xrightarrow{\quad\mathrm{pr}_3\quad} & \mathcal{C}at \\
{}_{\mathsf{s}}\mathsf{B} & \longleftarrow\!\shortmid & \big(({}_{\mathsf{s}}\mathsf{B})^* \leftarrow \mathsf{B} \to {}_{\mathsf{t}}\mathsf{B}\big) & \longmapsto & {}_{\mathsf{t}}\mathsf{B}
\end{array}
$$

Thus, ${}_{\mathsf{s}}\mathcal{SMQ}_{\mathcal{C}at} = {}_{\mathsf{t}}\mathcal{SMQ}_{\mathcal{C}at} = \mathcal{C}at$ and $\mathcal{SMQ}_{\mathcal{C}at}$ is an object of $\mathscr{S}$. Explicitly we may define objects, 1-morphisms and 2-morphisms of $\boxbox^{\mathbf{n}}\mathcal{SMQ}_{\mathcal{C}at}$ as $\operatorname{Ob}\boxbox^{\varnothing} = \mathcal{C}at$,

$$
\operatorname{Ob}\boxbox^{\mathbf{n}}\mathcal{SMQ}_{\mathcal{C}at} = \{(\mathcal{C}^*_{h-1} \xleftarrow{\mathrm{src}} \mathsf{C}_h \xrightarrow{\mathrm{tgt}} \mathcal{C}_h)_{h\in\mathbf{n}} \in \operatorname{Ob}\mathcal{SMQ}^n_{\mathcal{C}at}\}, \qquad n > 0,
$$

$$
\operatorname{Mor}\boxbox^{\varnothing} = \{F : \mathcal{C} \to \mathcal{D} - \text{functor }\},
$$

$\operatorname{Mor}\boxbox^{\mathbf{n}}\ \mathcal{SMQ}_{\mathcal{C}at}$

$$
= \left\{ \big((F_h : \mathsf{C}_h \to \mathsf{D}_h)_{h\in\mathbf{n}}, ({}_{\mathsf{t}}F_h : \mathcal{C}_h \to \mathcal{D}_h)_{h\in[n]}\big) \;\middle|\; \forall\, h \in \mathbf{n}\;
\begin{tikzcd}
\mathcal{C}^*_{h-1} \arrow[d, "{}_{\mathsf{t}}F^*_{h-1}"'] \arrow[dr, phantom, "="] & \mathsf{C}_h \arrow[l, "\mathrm{src}"'] \arrow[r, "\mathrm{tgt}"] \arrow[d, "F_h"] \arrow[dr, phantom, "="] & \mathcal{C}_h \arrow[d, "{}_{\mathsf{t}}F_h"] \\
\mathcal{D}^*_{h-1} & \mathsf{D}_h \arrow[l, "\mathrm{src}"] \arrow[r, "\mathrm{tgt}"'] & \mathcal{D}_h
\end{tikzcd}
\right\},
$$

$$
2\text{-}\operatorname{Mor}\boxbox^{\varnothing} = \{\phi : F \to G : \mathcal{C} \to \mathcal{D} - \text{natural transformation }\},
$$

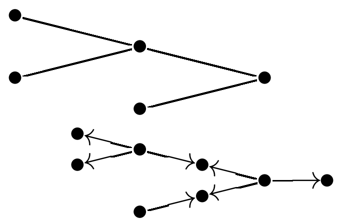
$$
2\text{-}\operatorname{Mor}\boxbox^{\mathbf{n}}\mathcal{SMQ}_{\mathcal{C}at} = \big\{\big((\phi_h : F_h \to G_h : \mathsf{C}_h \to \mathsf{D}_h)_{h\in\mathbf{n}}, ({}_{\mathsf{t}}\phi_h : {}_{\mathsf{t}}F_h \to {}_{\mathsf{t}}G_h : \mathcal{C}_h \to \mathcal{D}_h)_{h\in[n]}\big) \mid
\forall\, h \in \mathbf{n}\ ({}_{\mathsf{t}}\phi_{h-1}, \phi_h, {}_{\mathsf{t}}\phi_h) \text{ satisfy } (1.4)\big\}.
$$

2. A morphism $\boxdot^I : \boxbox^I\mathcal{SMQ}_{\mathcal{C}at} \to \mathcal{SMQ}_{\mathcal{C}at}$ of $\mathscr{S}$ for every set $I \in \operatorname{Ob}\mathcal{O}_{\mathsf{sk}}$. For $I = \varnothing$ the morphism $\boxdot^{\varnothing} : \boxbox^{\varnothing} = \mathcal{C}at \to \mathcal{SMQ}_{\mathcal{C}at}$, takes a category $\mathcal{C}$ to the $\mathcal{C}at$-span multiquiver $\mathsf{C} = \boxdot^{\varnothing}\mathcal{C}$ with ${}_{\mathsf{s}}\mathsf{C} = {}_{\mathsf{t}}\mathsf{C} = \mathcal{C}$, $\mathsf{C}(n) = \varnothing$ for $n \ne 1$, and $\mathcal{C} \xleftarrow{\mathrm{src}=\mathrm{id}} \mathsf{C}(1) = \mathcal{C} \xrightarrow{\mathrm{tgt}=\mathrm{id}} \mathcal{C}$. For $(\mathsf{C}_h)_{h\in I} = (\mathcal{C}^*_{h-1} \xleftarrow{\mathrm{src}} \mathsf{C}_h \xrightarrow{\mathrm{tgt}} \mathcal{C}_h)_{h\in I}$ with non-empty I there is a multiquiver defined as

$$
\boxdot^{h\in I}\mathsf{C}_h = \bigsqcup_{\text{tree } t:[I]\to\mathcal{O}_{\mathsf{sk}}} \lim(D_t : t^* \to \mathcal{C}at),
$$

where the diagram shape (category) t^* has $\operatorname{Ob} t^* = \mathrm{v}(t) \sqcup \overline{\mathrm{v}}(t) = \mathrm{v}(\bar t) \sqcup \mathrm{e}(\bar t)$. The tree $\bar t$ is obtained from the tree t by adding an extra edge starting from the root. The set $\mathrm{v}(\bar t)$ of internal vertices of $\bar t$ in the usual sense (without the new added root) coincides

with the set $\mathrm{v}(t)$ of internal vertices of t. The set $\mathrm{e}(\bar{t})$ of all edges of $\bar{t}$ is identified with the set $\overline{\mathrm{v}}(t)$ of all vertices of t. Arrows of the diagram start in an internal vertex of $\bar{t}$ and end up in (the middle of) an adjacent edge. Thus, $\operatorname{Mor} t^* = \{v \to e \mid (v,e) \in \mathrm{v}(\bar{t}) \times \mathrm{e}(\bar{t}),\ e \text{ is adjacent to } v\}$ is the set of flags of $\bar{t}$. Examples are given below:



$t =$, $t = \bullet$,

$t^* =$, $t^* = \bullet$.

The category assigned by D_t to $v = (h,b) \in \mathrm{v}(\bar{t})$ is $\mathsf{C}_h(t_h^{-1}b)$, the category assigned to $e_{(h-1,a)} : (h-1,a) \to (h,b) \in \mathrm{e}(\bar{t})$, $1 \leqslant h \leqslant I+1$, is $\mathcal{C}_{h-1}$. The functor assigned by D_t to the arrow $(h,b) \to e_{(h,b)} \in \operatorname{Mor} t^*$ is $\mathsf{C}_h(t_h^{-1}b) \xrightarrow{\mathrm{tgt}} \mathcal{C}_h$, $h \in I$. The functor assigned to the arrow $e_{(h-1,a)} \leftarrow (h,b) \in \operatorname{Mor} t^*$ for $a \in t_h^{-1}b$, $h \in I$, is $\mathcal{C}_{h-1} \xleftarrow{\mathrm{pr}_a} \mathcal{C}_{h-1}^{t_h^{-1}b} \xleftarrow{\mathrm{src}} \mathsf{C}_h(t_h^{-1}b)$. Explicitly we may describe the disjoint union of limits as

$$\boxdot^{h\in I}\mathsf{C}_h = \coprod_{\text{tree } t:[I]\to\mathcal{O}_{\mathsf{sk}}} \big\{(\ell,p) \mid \ell = (\ell_h^b)_{h\in[I]}^{b\in t(h)} \in \prod_{h\in[I]} \mathcal{C}_h^{t(h)},\ p = (p_h^b)_{h\in I}^{b\in t(h)} \in \prod_{h\in I}\prod_{b\in t(h)} \mathsf{C}_h(t_h^{-1}b),$$
$$\forall h\in I\ \forall b \in t(h)\ \ \mathrm{tgt}\, p_h^b = \ell_h^b,\ \mathrm{src}\, p_h^b = (\ell_{h-1}^a)_{a\in t_h^{-1}b}\big\}, \quad (1.6)$$

where elements of a category mean its morphisms and, in particular, its objects. The source and the target functors are given by $\mathrm{src} : \boxdot^{h\in I}\mathsf{C}_h \to \mathcal{C}_0^*$, $\mathrm{src}(\ell,p) = (\ell_0^a)_{a\in t(0)}$ and $\mathrm{tgt} : \boxdot^{h\in I}\mathsf{C}_h \to \mathcal{C}_{\max[I]}$, $\mathrm{tgt}(\ell,p) = \ell_{\max[I]}^1$.

3. For any non-decreasing map $\phi : \mathbf{n} \to \mathbf{k}$ and the induced $\psi = [\phi] : [k] \to [n]$ (see (0.12)) there are 2-isomorphisms

$$\Lambda^\phi_{\mathcal{SMQ}_{\mathcal{C}at}} : \boxdot^{l\in\mathbf{n}}\mathsf{C}_l \longrightarrow \boxdot^{m\in\mathbf{k}}\boxdot^{l\in\phi^{-1}m}\mathsf{C}_l,$$
$$(t,\ell,(p_l^j)_{l\in\mathbf{n}}^{j\in t(l)}) \longmapsto (t_\psi,\ell_\psi,(t^{|j}_{[\psi(m-1),\psi(m)]},\ell^{|j},(p_l^i)_{l\in\phi^{-1}m}^{i\in t^{|j}_{[\psi(m-1),\psi(m)]}(l)})_{m\in\mathbf{k}}^{j\in t(\psi(m))}).$$

4. The 2-isomorphism $\mathrm{P} : \boxdot^{\mathbf{1}} \to \mathrm{Iso} : \boxdot^{\mathbf{1}}\mathcal{SMQ}_{\mathcal{C}at} \xrightarrow{\sim} \mathcal{SMQ}_{\mathcal{C}at}$ is the obvious one.

Fix a category $\mathcal{C}$. Consider the $\mathcal{C}at$-subcategory ${}^{\mathcal{C}}\mathcal{SMQ}_{\mathcal{C}at}$ of the $\mathcal{C}at$-category $\mathcal{SMQ}_{\mathcal{C}at}$ whose objects are $\mathcal{C}at$-span multiquivers C with ${}_{\mathsf{s}}\mathsf{C} = {}_{\mathsf{t}}\mathsf{C} = \mathcal{C}$, whose 1-morphisms satisfy ${}_{\mathsf{s}}F = {}_{\mathsf{t}}F = \mathrm{id}_{\mathcal{C}}$ and 2-morphisms satisfy ${}_{\mathsf{s}}\phi = {}_{\mathsf{t}}\phi = \mathrm{id}_{\mathrm{Id}}$.

1.5 Definition. A *lax $\mathcal{C}at$-span multicategory* C is the collection of

1. a $\mathcal{C}at$-span multiquiver $\mathsf{C} = \big(\mathcal{C}^* \xleftarrow{\mathrm{src}} \mathsf{C} \xrightarrow{\mathrm{tgt}} \mathcal{C}\big)$;

2. a 1-morphism $\circledast^I : \boxdot^I\mathsf{C} \to \mathsf{C}$ in ${}^{\mathcal{C}}\mathcal{SMQ}_{\mathcal{C}at}$, for every set $I \in \operatorname{Ob}\mathcal{O}_{\mathsf{sk}}$.

   For a map $f : I \to J$ in $\operatorname{Mor}\mathcal{O}_{\mathsf{sk}}$ introduce a 1-morphism

$$\circledast^f_{\mathsf{C}} = \big(\boxdot^I\mathsf{C} \xrightarrow{\Lambda^f} \boxdot^{j\in J}\boxdot^{f^{-1}j}\mathsf{C} \xrightarrow{\boxdot^{j\in J}\circledast^{f^{-1}j}} \boxdot^J\mathsf{C}\big);$$

3. a 2-morphism λ^f in ${}^{\mathcal{C}}\mathcal{SMQ}_{\mathcal{C}at}$ for every map $f : I \to J$ in $\operatorname{Mor} \mathcal{O}_{\mathsf{sk}}$:

$$
\begin{tikzcd}
\boxdot^I \mathsf{C} \arrow[r, "\circledast^f_{\mathsf{C}}"] \arrow[dr, "\circledast^I"'] \arrow[r, phantom, shift right=4ex, "\Uparrow \lambda^f"] & \boxdot^J \mathsf{C} \arrow[d, "\circledast^J"] \\
& \mathsf{C}
\end{tikzcd}
=
\begin{tikzcd}
\boxdot^{j\in J} \boxdot^{f^{-1}j} \mathsf{C} \arrow[r, "\boxdot^{j\in J} \circledast^{f^{-1}j}"] & \boxdot^J \mathsf{C} \arrow[d, "\circledast^J"] \\
\boxdot^I \mathsf{C} \arrow[u, "\Lambda^f"] \arrow[r, "\circledast^I"'] \arrow[ur, phantom, "\Uparrow \lambda^f"] & \mathsf{C}
\end{tikzcd}
$$

4. a 2-morphism $\rho : \circledast^{\mathbf{1}} \to \mathrm{P} : \boxdot^{\mathbf{1}} \mathsf{C} \to \mathsf{C}$ in ${}^{\mathcal{C}}\mathcal{SMQ}_{\mathcal{C}at}$,

such that

(i) for all sets $I \in \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}$

$$
\begin{tikzcd}
\boxdot^{i\in I} \boxdot^{\{i\}} \mathsf{C} \arrow[r, shift left=1.5ex, "\boxdot^{i\in I} \mathrm{P}^{\{i\}}"] \arrow[r, shift right=1.5ex, "\boxdot^{i\in I} \circledast^{\{i\}}"'] \arrow[r, phantom, "\Uparrow \boxdot^{i\in I} \rho^{\{i\}}"] & \boxdot^I \mathsf{C} \arrow[d, "\circledast^I"] \\
\boxdot^I \mathsf{C} \arrow[u, "\Lambda^{\mathrm{id}_I}"] \arrow[r, "\circledast^I"'] \arrow[ur, phantom, "\Uparrow \lambda^{\mathrm{id}_I}"] & \mathsf{C}
\end{tikzcd}
=
\begin{tikzcd}
\boxdot^{i\in I} \boxdot^{\{i\}} \mathsf{C} \arrow[r, "\boxdot^{i\in I} \mathrm{P}^{\{i\}}"] & \boxdot^I \mathsf{C} \arrow[d, "\circledast^I"] \\
\boxdot^I \mathsf{C} \arrow[u, "\Lambda^{\mathrm{id}_I}"] \arrow[ur, "1"] \arrow[r, "\circledast^I"'] & \mathsf{C}
\end{tikzcd}
\equiv \mathrm{id} : \circledast^I \to \circledast^I,
$$

$$
\begin{tikzcd}
\boxdot^{\mathbf{1}} \boxdot^I \mathsf{C} \arrow[r, "\boxdot^{\mathbf{1}} \circledast^I"] & \boxdot^{\mathbf{1}} \mathsf{C} \arrow[d, shift right=1.5ex, "\circledast^{\mathbf{1}}"'] \arrow[d, shift left=1.5ex, "\mathrm{P}"] \arrow[d, phantom, "\overset{\rho}{\Rightarrow}"] \\
\boxdot^I \mathsf{C} \arrow[u, "\Lambda^{I\to\mathbf{1}}"] \arrow[r, "\circledast^I"'] \arrow[ur, phantom, "\Uparrow \lambda^{I\to\mathbf{1}}"] & \mathsf{C}
\end{tikzcd}
=
\begin{tikzcd}
\boxdot^{\mathbf{1}} \boxdot^I \mathsf{C} \arrow[rr, "\boxdot^{\mathbf{1}} \circledast^I"] \arrow[dr, "\mathrm{P}"] & & \boxdot^{\mathbf{1}} \mathsf{C} \arrow[d, "\mathrm{P}"] \\
\boxdot^I \mathsf{C} \arrow[u, "\Lambda^{I\to\mathbf{1}}"] \arrow[r, "1"'] & \boxdot^I \mathsf{C} \arrow[r, "\circledast^I"'] & \mathsf{C}
\end{tikzcd}
\equiv \mathrm{id} : \circledast^I \to \circledast^I;
$$

(ii) for any pair of composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ from $\mathcal{O}_{\mathsf{sk}}$ this equation holds:

$$
\begin{tikzcd}
\boxdot^J \mathsf{C} \arrow[r, "\circledast^g_{\mathsf{C}}"] \arrow[dr, "\circledast^J"] \arrow[r, phantom, shift right=5ex, "\Uparrow \lambda^g"] & \boxdot^K \mathsf{C} \arrow[d, "\circledast^K"] \\
\boxdot^I \mathsf{C} \arrow[u, "\circledast^f_{\mathsf{C}}"] \arrow[r, "\circledast^I"'] \arrow[u, phantom, shift left=5ex, "\Uparrow \lambda^f"] & \mathsf{C}
\end{tikzcd}
=
\begin{tikzcd}
\boxdot^J \mathsf{C} \arrow[r, "\circledast^g_{\mathsf{C}}"] \arrow[r, phantom, shift right=5ex, "\Uparrow \boxdot^{k\in K} \lambda^{f : f^{-1}g^{-1}k \to g^{-1}k}"] & \boxdot^K \mathsf{C} \arrow[d, "\circledast^K"] \\
\boxdot^I \mathsf{C} \arrow[u, "\circledast^f_{\mathsf{C}}"] \arrow[ur, "\circledast^{g\circ f}_{\mathsf{C}}"] \arrow[r, "\circledast^I"'] \arrow[r, phantom, shift left=5ex, "\Uparrow \lambda^{g\circ f}"] & \mathsf{C}
\end{tikzcd}
$$

A *weak* $\mathcal{C}at$*-span multicategory* C is a lax $\mathcal{C}at$-span multicategory C such that λ^f, ρ are invertible.

Here $\boxdot^{k\in K}\lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}$ means the 2-morphism

$$
\begin{tikzcd}
 & & \boxdot^{J}\mathsf{C} \arrow[drr, "\Lambda^g"] \arrow[d, phantom, "="] & & \\
\boxdot^{j\in J}\boxdot^{f^{-1}j}\mathsf{C} \arrow[urr, "\boxdot^{j\in J}\circledast^{f^{-1}j}"] \arrow[rr, "\Lambda^g"] & & \boxdot^{k\in K}\boxdot^{j\in g^{-1}k}\boxdot^{f^{-1}j}\mathsf{C} \arrow[rr, "\boxdot^{k\in K}\boxdot^{j\in g^{-1}k}\circledast^{f^{-1}j}"] & & \boxdot^{k\in K}\boxdot^{g^{-1}k}\mathsf{C} \arrow[d, "\boxdot^{k\in K}\circledast^{g^{-1}k}"] \\
\boxdot^{I}\mathsf{C} \arrow[u, "\Lambda^f"] \arrow[urr, phantom, "="] \arrow[rr, "\Lambda^{g\circ f}"'] & & \boxdot^{k\in K}\boxdot^{f^{-1}g^{-1}k}\mathsf{C} \arrow[u, "\boxdot^{k\in K}\Lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}"] \arrow[urr, phantom, "\Uparrow\,\boxdot^{k\in K}\lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}"] \arrow[rr, "\boxdot^{k\in K}\circledast^{f^{-1}g^{-1}k}"'] & & \boxdot^{K}\mathsf{C}
\end{tikzcd}
$$

The top quadrilateral in above diagram is the identity 2-morphism due to the 2-transformation Λ^g being strict. The left square is the tetrahedron equation for Λ.

1.6 Examples. 1) Assume that $\mathsf{C} = \big(\mathcal{C} \xleftarrow{\text{src}} \mathsf{C} \xrightarrow{\text{tgt}} \mathcal{C}\big)$ is a lax $\mathcal{C}at$-span multicategory. In particular, we assume that $\mathsf{C}(n) = \varnothing$ for $n \neq 1$. Then there is only one summand in (1.6) – the one indexed by the linear tree $t : [I] \to \mathcal{O}_{\mathsf{sk}}$, $t(i) = \mathbf{1}$. Thus,

$$\boxdot^{\mathbf{m}}\mathsf{C} = \lim(\mathcal{C} \xleftarrow{\text{src}} \mathsf{C} \xrightarrow{\text{tgt}} \mathcal{C} \xleftarrow{\text{src}} \mathsf{C} \xrightarrow{\text{tgt}} \dots \mathcal{C} \xleftarrow{\text{src}} \mathsf{C} \xrightarrow{\text{tgt}} \mathcal{C}), \tag{1.7}$$

where the number of vertices C is m and the limit is taken in $\mathcal{C}at$. In these assumptions we describe Examples 1.1–1.3:

1.1) Denote by $\mathbb{1}$ the terminal category. A lax $\mathcal{C}at$-span multicategory $\mathbb{1} \xleftarrow{\text{src}} \mathsf{C} \xrightarrow{\text{tgt}} \mathbb{1}$ is nothing else but a lax Monoidal category, see [BLM08, Definition 2.5]. In fact, here $\boxdot^I\mathsf{C} = \mathsf{C}^I$ and $\circledast^I : \mathsf{C}^I \to \mathsf{C}$ becomes the Monoidal product.

1.2) A strict $\mathcal{C}at$-span multicategory $(\mathcal{C} \xleftarrow{\text{src}} \mathsf{C} \xrightarrow{\text{tgt}} \mathcal{C}, \circledast^I, \lambda^f = \mathrm{id}, \rho = \mathrm{id})$ is the same thing as a category internal to $\mathcal{C}at$. Equivalently, it is a double category and $\mathcal{C}$ is its category of vertical morphisms.

1.3) A lax $\mathcal{C}at$-span multicategory $(\mathcal{C} \xleftarrow{\text{src}} \mathsf{C} \xrightarrow{\text{tgt}} \mathcal{C}, \circledast^I, \lambda^f, \rho)$ can be called a lax double category. Fix an object A of $\mathcal{C}$ and consider the subcategory D of C, whose objects X satisfy $\operatorname{src} X = \operatorname{tgt} X = A$ and morphisms f satisfy $\operatorname{src} f = \operatorname{tgt} f = 1_A$. Then $\mathbb{1} = \{A\} \leftarrow \mathsf{D} \rightarrow \{A\} = \mathbb{1}$ is a lax $\mathcal{C}at$-span multicategory of the type considered in Example 1.1. Thus, $(\mathsf{D}, \circledast^I, \lambda^f, \rho)$ is a lax Monoidal category.

2) Assume that C is a lax $\mathcal{C}at$-span multicategory, $\mathcal{C}$, C are strict 2-categories, $\mathcal{C} \xleftarrow{\text{src}} \mathsf{C} \xrightarrow{\text{tgt}} \mathcal{C}$ and $\circledast^I : \boxdot^I\mathsf{C} \to \mathsf{C}$ are strict 2-functors, λ^f and ρ are strict 2-morphisms. In other words, all data are enriched in $\mathcal{C}at$. Notice as above that $\mathsf{C}(n) = \varnothing$ for $n \neq 1$. In these assumptions we have Examples 2.1–2.3:

2.1) If $\mathcal{C} = \mathbb{1}$ is the terminal 2-category (that with a unique 2-morphism), then C is nothing else but a lax Monoidal $\mathcal{C}at$-category, see [BLM08, Definition 2.10].

2.2) If $\lambda^f = \mathrm{id}$, $\rho = \mathrm{id}$, then C is the same thing as an internal category in strict-2-$\mathcal{C}at$ = $\mathcal{C}at$-$\mathcal{C}at$. Equivalently, it is a triple category, whose one direction category is discrete. Its 3-cells can be visualized as cylinders.

2.3) Dropping the restrictions of 2.2) we may call C a lax triple category, whose one direction category is discrete. For instance, $\mathfrak{SMQ}_{\mathcal{C}at}$ is such a weak triple category. Notice that its powers are given by $\boxdot^{\mathbf{n}}$ in place of $\boxdot^{\mathbf{n}}$, compare (1.5) and (1.7). Also $(\mathfrak{SMQ}_{\mathcal{C}at}, \boxdot^I, \Lambda^{\phi}_{\mathfrak{SMQ}_{\mathcal{C}at}}, \mathrm{P})$ uses $\boxdot^I$, $\Lambda^{\phi}_{\mathfrak{SMQ}_{\mathcal{C}at}}$, P in place of $\circledast^I$, λ^ϕ, ρ.

Fix an object A of $\mathcal{C}$ and consider the 2-subcategory D of C, whose objects X (resp. 1-morphisms f, 2-morphisms α) satisfy $\operatorname{src} X = \operatorname{tgt} X = A$ (resp. $\operatorname{src} f = \operatorname{tgt} f = 1_A$, $\operatorname{src}\alpha = \operatorname{tgt}\alpha = 1_{1_A}$). Then $\mathbb{1} = \{A\} \leftarrow \mathsf{D} \to \{A\} = \mathbb{1}$ is a lax $\mathcal{C}at$-span submulticategory of the type considered in Example 2.1. Thus, $(\mathsf{D}, \circledast^I, \lambda^\phi, \rho)$ is a lax Monoidal $\mathcal{C}at$-category. In particular, $({}^{\mathcal{C}}\mathfrak{SMQ}_{\mathcal{C}at}, \boxdot^I, \Lambda^{\phi}_{\mathfrak{SMQ}_{\mathcal{C}at}}, \mathrm{P})$ is a (weak) Monoidal $\mathcal{C}at$-category. Now we may say that *a lax $\mathcal{C}at$-span multicategory $(\mathsf{C}, \circledast^I, \lambda^\phi, \rho)$ is a lax-Monoidal-category inside the Monoidal $\mathcal{C}at$-category* $({}^{\mathcal{C}}\mathfrak{SMQ}_{\mathcal{C}at}, \boxdot^I, \Lambda^{\phi}_{\mathfrak{SMQ}_{\mathcal{C}at}}, \mathrm{P})$.

3) A strict $\mathcal{C}at$-span multicategory C (the one with $\lambda^\phi = \mathrm{id}$, $\rho = \mathrm{id}$) is the same thing as a *-multicategory in the sense of Leinster [Lei03, Definition 4.2.2] for the Cartesian monad $\text{-}^* : \mathcal{C}at \to \mathcal{C}at$, $\mathcal{C} \mapsto \mathcal{C}^*$ of free strict monoidal category.

4) Let us discuss also the particular case of a $\mathcal{C}at$-span multiquiver C for which ${}_{\mathsf{s}}\mathsf{C}$ and ${}_{\mathsf{t}}\mathsf{C}$ are discrete categories. In that case the category C is a disjoint union of full subcategories $\mathsf{C}((X_i)_{i\in I}; Y)$, $X_i \in \operatorname{Ob}{}_{\mathsf{s}}\mathsf{C}$, $Y \in \operatorname{Ob}{}_{\mathsf{t}}\mathsf{C}$. Therefore, the notion of a $\mathcal{C}at$-span multiquiver coincides in the mentioned case with the notion of a $\mathcal{C}at$-multiquiver [BLM08, Definition 3.2]. The latter notion is a particular case of $\mathcal{V}$-multiquivers defined for an arbitrary monoidal category $\mathcal{V}$, not only for $\mathcal{V} = \mathcal{C}at$.

On the other hand, the notion a lax $\mathcal{C}at$-span multicategory C with the discrete category of objects ${}_{\mathsf{t}}\mathsf{C}$, called shortly a lax $\mathcal{C}at$-multicategory, comprises more examples than the notion a $\mathcal{C}at$-multicategory (a particular case of $\mathcal{V}$-multicategories [BLM08, Definition 3.7]). In fact, natural transformations λ^f and ρ have to be identity transformations in the latter case.

1.7. Lax $\mathcal{C}at$-span operads. A *lax $\mathcal{C}at$-span operad* is a particular case of a lax $\mathcal{C}at$-span multicategory C – that for which ${}_{\mathsf{t}}\mathsf{C} = \mathbb{1}$ is the terminal category. Lax $\mathcal{C}at$-operad is a shorthand for the lax $\mathcal{C}at$-span operad. Let us give more comments on its structure. First of all, the monoidal product of multiquivers C_h in the category ${}^{\mathbb{1}}\mathfrak{SMQ}_{\mathcal{C}at}$ is

$$\boxdot^{h\in I}\mathsf{C}_h = \coprod_{\text{tree } t:[I]\to\mathcal{O}_{\mathsf{sk}}} \prod_{h\in I} (\mathsf{C}_h)^{t(h)},$$

$$(\boxdot^{h\in I}\mathsf{C}_h)(n) = \coprod_{\text{tree } t:[I]\to\mathcal{O}_{\mathsf{sk}}}^{|t(0)|=n} \prod_{h\in I}\prod_{b\in t(h)} \mathsf{C}_h(t_h^{-1}b).$$

The 1-morphism $\circledast^I : \boxdot^I\mathsf{C} \to \mathsf{C}$ takes the form of a collection of functors

$$\circledast^I(n) : \coprod_{\text{tree } t:[I]\to\mathcal{O}_{\mathsf{sk}}}^{|t(0)|=n} \prod_{h\in I}\prod_{b\in t(h)} \mathsf{C}_h(t_h^{-1}b) \to \mathsf{C}(n). \tag{1.8}$$

1.8 Proposition. *Let C be a lax $\mathcal{C}at$-span operad. This induces a Monoidal structure on the category* $\mathsf{C}(1)$.

Proof. Define a *linear tree* corresponding to a set $I \in \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}$ as the functor $lt_I : [I] \to \mathcal{O}_{\mathsf{sk}}$, $[I] \ni i \mapsto \mathbf{1}$. We may view $lt_I = (\mathbf{1} \to \mathbf{1} \to \cdots \to \mathbf{1})$ as a synonym for $[I]$. Restricting functor (1.8) to the linear tree lt_I we get a functor

$$\odot^I \overset{\text{def}}{=} \circledast^I(lt_I)(1)| : \mathsf{C}(1)^I \to \mathsf{C}(1).$$

Notice that for a linear tree t and any $f : I \to J$ the trees t_ψ and $t^{|1}_{[\psi(m-1),\psi(m)]}$ from Section 0.27.1 are also linear. Therefore, $\circledast^f(1)$ maps the component indexed by the linear tree lt_I to the component indexed by lt_J, and this restriction $\circledast^f(1)|$ coincides with $\odot^f$.

Restricting given 2-morphism λ^f (for $\boxdot$) to linear trees we get a natural transformation $\lambda^f_\odot$ (for $\odot$). The given 2-morphism $\rho : \circledast^{\mathbf{1}} \to \mathrm{P} : \boxdot^{\mathbf{1}}\mathsf{C} \to \mathsf{C}$ restricted to the linear tree $lt_{\mathbb{1}} = (\mathbf{1} \to \mathbf{1})$ gives a transformation $\rho_\odot : \odot^{\mathbf{1}} \to \mathrm{Iso} : \mathsf{C}(1)^{\mathbf{1}} \to \mathsf{C}(1)$.

Two equations for $\lambda^f_\odot$ and $\rho_\odot$ follow from two equations 1.5(i) for λ^f_{C} and ρ_{C}. The tetrahedron equation for $\lambda^f_\odot$ follows from tetrahedron equation 1.5(ii) for λ^f_{C}. $\square$

1.9. Lax $\mathcal{C}at$-operad of graded $\Bbbk$-modules. Let us construct examples G, DG of a weak $\mathcal{C}at$-operad: that of (differential) graded $\Bbbk$-modules. An operad is a multicategory with one object $*$. Instead of $\mathsf{G}({}^I*;*)$, $I \in \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}$, the notation $\mathsf{G}(I)$ is used. We define $\mathsf{G}(I) = \mathbf{gr}^{\mathbb{N}^I}$, $\mathsf{DG}(I) = \mathbf{dg}^{\mathbb{N}^I}$. The two cases are similar, differing only by absence or presence of the differential. So we give our formulae only for one of them. A tree t gives rise to a functor

$$\circledast(t) : \prod_{(h,b)\in \mathrm{v}(t)} \mathsf{G}(t_h^{-1}b) \to \mathsf{G}(t(0)), \qquad (\mathcal{P}^b_h)_{(h,b)\in \mathrm{v}(t)} \mapsto \circledast(t)(\mathcal{P}^b_h)_{(h,b)\in \mathrm{v}(t)}.$$

A t-tree is a functor $\tau : t \to \mathcal{O}_{\mathsf{sk}}$, $\tau(\mathrm{root}) = \mathbf{1}$, where the poset t is the free category built on the quiver t oriented from the root. It has the set of objects $\overline{\mathrm{v}}(t)$, the set of vertices of t, morphisms are oriented paths, and the root is the terminal object of t.

A tree can be defined as a successor map $S : V \to V$, where V is a non-empty finite set (of vertices), such that $\operatorname{Im}(S^k)$ contains only one element for some $k \in \mathbb{N}$. There is only one vertex $v \in V$ such that $S(v) = v$, it is called the root. An oriented graph without loops G is constructed out of S, whose set of vertices is V and arrows are $v \to S(v)$ if vertex v is not the root. Since G is a connected graph, whose number of edges is one less than the number of vertices, it is a tree. The only oriented path connecting a vertex v with the root consists of v, $S(v)$, $S^2(v)$, ..., the root. For any tree T denote by $S_T : \overline{\mathrm{v}}(T) \to \overline{\mathrm{v}}(T)$ its successor map. A morphism of $S : V \to V$ to $S' : V' \to V'$ is a mapping $f : V \to V'$ such that $f \cdot S' = S \cdot f$. A t-tree τ comes with a morphism $S_\tau \to S_t$ of successor maps.

Thus, for a tree $t : [I] \to \mathcal{O}_{\mathsf{sk}}$ with $I = \mathbf{n}$, $[I] = [n] = \{0, 1, \dots, n\}$,

$$\circledast(t)(\mathcal{P}^b_h)_{(h,b)\in \mathrm{v}(t)}(z) = \bigoplus^{t-\text{tree } \tau}_{\forall a\in t(0)\, |\tau(0,a)|=z^a} \bigotimes^{h\in I} \bigotimes^{b\in t(h)} \bigotimes^{p\in\tau(h,b)} \mathcal{P}^b_h\Big(\big(|\tau^{-1}_{(h-1,a)\to(h,b)}(p)|\big)_{a\in t_h^{-1}b}\Big). \tag{1.9}$$

The following 2-isomorphism has to be specified for a non-decreasing map $f : I \to J$, a tree t of height $|I|$ and the induced map $\psi = [f] : [J] \to [I]$ and trees t_ψ, $t^{|c}_{[\psi(g-1),\psi(g)]}$ (see (0.12))

$$\begin{array}{ccc}
\prod_{(h,b)\in\mathrm{v}(t)} \mathsf{DG}(t_h^{-1}b) & \xrightarrow[\sim]{\Lambda^f} & \prod_{(g,c)\in\mathrm{v}(t_\psi)} \prod_{(h,b)\in\mathrm{v}(t^{|c}_{[\psi(g-1),\psi(g)]})} \mathsf{DG}(t_h^{-1}b) \\
{\scriptstyle\circledast(t)}\Big\downarrow & \xRightarrow{\lambda^f} & \Big\downarrow \prod_{(g,c)\in\mathrm{v}(t_\psi)} \circledast(t^{|c}_{[\psi(g-1),\psi(g)]}) \\
\mathsf{DG}(t(0)) & \xleftarrow{\circledast(t_\psi)} & \prod_{(g,c)\in\mathrm{v}(t_\psi)} \mathsf{DG}((t_\psi)_g^{-1}c)
\end{array}$$

This is an invertible natural transformation

$$\begin{array}{ccc}
\prod_{(h,b)\in\mathrm{v}(t)} \mathbf{dg}^{\mathbb{N}^{t_h^{-1}b}} & \xrightarrow[\sim]{\Lambda^f} & \prod_{(g,c)\in\mathrm{v}(t_\psi)} \prod_{(h,b)\in\mathrm{v}(t^{|c}_{[\psi(g-1),\psi(g)]})} \mathbf{dg}^{\mathbb{N}^{t_h^{-1}b}} \\
{\scriptstyle\circledast(t)}\Big\downarrow & \xRightarrow{\lambda^f} & \Big\downarrow \prod_{(g,c)\in\mathrm{v}(t_\psi)} \circledast(t^{|c}_{[\psi(g-1),\psi(g)]}) \\
\mathbf{dg}^{\mathbb{N}^{t(0)}} & \xleftarrow{\circledast(t_\psi)} & \prod_{(g,c)\in\mathrm{v}(t_\psi)} \mathbf{dg}^{\mathbb{N}^{(t_\psi)_g^{-1}c}}
\end{array}$$

On collections a natural bijection has to be constructed:

$$\lambda^f : \circledast(t)(\mathcal{P}^b_h)_{(h,b)\in\mathrm{v}(t)} \to \circledast(t_\psi)\big(\circledast(t^{|c}_{[\psi(g-1),\psi(g)]})(\mathcal{P}^b_h)_{(h,b)\in\mathrm{v}(t^{|c}_{[\psi(g-1),\psi(g)]})}\big)_{(g,c)\in\mathrm{v}(t_\psi)}.$$

Denote

$$\mathcal{Q}^c_g = \circledast(t^{|c}_{[\psi(g-1),\psi(g)]})(\mathcal{P}^b_h)_{(h,b)\in\mathrm{v}(t^{|c}_{[\psi(g-1),\psi(g)]})}.$$

For an arbitrary t-tree τ define a t_ψ-tree ${}_\psi\tau$ as the composition $t_\psi \to t \xrightarrow{\tau} \mathcal{O}_{\mathsf{sk}}$; the first functor being constructed from the maps $\mathrm{id} : t_\psi(g) \to t(\psi(g))$. For arbitrary $g \in J$, $c \in t_\psi(g)$, $q \in \tau(\psi(g), c)$ define a tree ${}^{\,q}_{g,c}\tau : t^{|c}_{[\psi(g-1),\psi(g)]} \to \mathcal{O}_{\mathsf{sk}}$ with ${}^{\,q}_{g,c}\tau(h,b) \simeq \tau^{-1}_{(h,b)\to(\psi(g),c)}(q)$.

Define λ^f as the composition

$$\circledast(t)(\mathcal{P}_h^b)_{(h,b)\in\mathrm{v}(t)}(z) = \bigoplus_{\forall a\in t(0)\ |\tau(0,a)|=z^a}^{t-\text{tree }\tau} \bigotimes^{h\in I} \bigotimes^{b\in t(h)} \bigotimes^{p\in\tau(h,b)} \mathcal{P}_h^b\Big(\big(|\tau^{-1}_{a\to b}(p)|\big)_{a\in t_h^{-1}b}\Big)$$

$$\xrightarrow{\sim} \bigoplus_{\substack{\forall a\in t(0)\\ |_\psi\tau(0,a)|=z^a}}^{t_\psi-\text{tree }_\psi\tau} \bigoplus_{\substack{\forall d\in t^{-1}_{\psi,g}(c)\\ {}_{g,c}^{\;q}\tau(\psi(g-1),d)\simeq{}_\psi\tau^{-1}_{d\to c}(q)}}^{\substack{\forall g\in J,\, c\in t_\psi(g),\, q\in{}_\psi\tau(g,c)\\ t^{|c}_{[\psi(g-1),\psi(g)]}-\text{tree }{}_{g,c}^{\;q}\tau}} \bigotimes_{q\in{}_\psi\tau(g,c)}^{\substack{g\in J\\ c\in t_\psi(g)}} \bigotimes_{h>\psi(g-1)}^{\substack{h\in J\\ h\leqslant\psi(g)}} \bigotimes_{p\in{}_{g,c}^{\;q}\tau(h,b)}^{b\in t^{-1}_{h\to\psi(g)}(c)} \mathcal{P}_h^b\Big(\big(|{}_{g,c}^{\;q}\tau^{-1}_{a\to b}(p)|\big)_{a\in t_h^{-1}b}\Big)$$

$$\xrightarrow{\sim} \bigoplus_{\substack{\forall a\in t(0)\\ |_\psi\tau(0,a)|=z^a}}^{t_\psi-\text{tree }_\psi\tau} \bigotimes_{\substack{c\in t_\psi(g)\\ q\in{}_\psi\tau(g,c)}}^{g\in J} \bigoplus_{\substack{\forall d\in t^{-1}_{\psi,g}(c)\\ {}_{g,c}^{\;q}\tau(\psi(g-1),d)\simeq{}_\psi\tau^{-1}_{d\to c}(q)}}^{t^{|c}_{[\psi(g-1),\psi(g)]}-\text{tree }{}_{g,c}^{\;q}\tau} \bigotimes_{\substack{h\leqslant\psi(g)\\ h>\psi(g-1)}}^{h\in J} \bigotimes_{p\in{}_{g,c}^{\;q}\tau(h,b)}^{b\in t^{-1}_{h\to\psi(g)}(c)} \mathcal{P}_h^b\Big(\big(|{}_{g,c}^{\;q}\tau^{-1}_{a\to b}(p)|\big)_{a\in t_h^{-1}b}\Big)$$

$$= \bigoplus_{\forall a\in t(0)\ |_\psi\tau(0,a)|=z^a}^{t_\psi-\text{tree }_\psi\tau} \bigotimes^{g\in J} \bigotimes^{c\in t_\psi(g)} \bigotimes^{q\in{}_\psi\tau(g,c)} \mathcal{Q}_g^c\Big(\big(|{}_\psi\tau^{-1}_{d\to c}(q)|\big)_{d\in t^{-1}_{\psi,g}c}\Big)$$

$$= \circledast(t_\psi)\Big(\circledast(t^{|c}_{[\psi(g-1),\psi(g)]})(\mathcal{P}_h^b)_{(h,b)\in\mathrm{v}(t^{|c}_{[\psi(g-1),\psi(g)]})}\Big)_{(g,c)\in\mathrm{v}(t_\psi)}. \tag{1.10}$$

The second isomorphism is the distributivity isomorphism between the tensor product and the direct sum. The first mapping is well defined, since for any t-tree τ the trees constructed out of it satisfy

$${}_{g,c}^{\;q}\tau(\psi(g-1),d) \simeq \tau^{-1}_{(\psi(g-1),d)\to(\psi(g),c)}(q) = {}_\psi\tau^{-1}_{(g-1,d)\to(g,c)}(q).$$

Thus τ is mapped to the set of collections of trees of the following type

$$\big({}_\psi\tau, {}_{g,c}^{\;q}\tau \mid g\in J,\, c\in t_\psi(g),\, q\in{}_\psi\tau(g,c)\ \forall d\in t^{-1}_{\psi,g}(c)\ {}_{g,c}^{\;q}\tau(\psi(g-1),d)\simeq{}_\psi\tau^{-1}_{(g-1,d)\to(g,c)}(q)\big). \tag{1.11}$$

The last condition means that we are given a morphism ${}_{g,c}^{\;q}\tau(\psi(g-1),d)\to{}_\psi\tau(g-1,d)\in\mathcal{O}_{\mathsf{sk}}$, whose image in $\mathcal{S}et$ coincides with the preimage of q under the map ${}_\psi\tau_{(g-1,d)\to(g,c)}:{}_\psi\tau(g-1,d)\to{}_\psi\tau(g,c)$. This gives a bijection from the set of trees τ to the set of such collections. The inverse mapping is given by the formula

$$\tau(h,b) = \bigsqcup_{q\in{}_\psi\tau(\phi h,t_{h\to\psi\phi h}(b))} {}_{\phi h,t_{h\to\psi\phi h}(b)}^{\qquad q}\tau(h,b), \tag{1.12}$$

where the disjoint union is taken in the sense of $\mathcal{O}_{\mathsf{sk}}$.

Disjoint union of a family of sets $(S_k)_{k\in K}$ is

$$\bigsqcup_{k\in K} S_k = \big\{(k,s)\in K\times\bigcup_{k\in K}S_k \mid s\in S_k\big\}. \tag{1.13}$$

If K and all S_k are ordered, then $\sqcup_{k\in K}S_k$ is lexicographically ordered, $k_1 < k_2$ implies $(k_1, s_1) < (k_2, s_2)$, and $(k, s) \leqslant (k, s')$ is equivalent to $s \leqslant s'$. When K and all S_k are totally ordered, then so is the disjoint union. If K and all S_k are objects of $\mathcal{O}_{\mathsf{sk}}$, we denote by $\sqcup_{k\in K}S_k$ the only object of $\mathcal{O}_{\mathsf{sk}}$ in bijection with (1.13). It comes equipped with morphisms of $\mathcal{O}_{\mathsf{sk}}$

$$S_k \xrightarrow{\mathrm{in}_k} \sqcup_{k\in K} S_k \longrightarrow K.$$

We may view (1.13) as an interpretation (a generalization) of a sum $\sum_{k\in K} s_k$ of non-negative integers.

Let us define the successor map $\tau_{(h-1,b)\to(h,t_h(b))} : \tau(h-1, b) \to \tau(h, t_h(b))$ for the tree τ constructed out from the collection $({}_\psi\tau, {}_{g,c}^{\ q}\tau)$ in (1.12). Assume first that $\phi(h-1) = \phi(h)$. Then $\psi\phi(h-1) = \psi\phi(h)$, the indexing sets coincide, and the sought successor map is a disjoint union of successor maps for each summand:

$$\tau_{(h-1,b)\to(h,t_h(b))} = \bigsqcup_{\mathrm{id}} {}_{\phi h, t_{h-1\to\psi\phi h}(b)}^{\qquad q}\tau_{(h-1,b)\to(h,t_h(b))} :$$

$$\tau(h-1, b) = \bigsqcup_{q\in{}_\psi\tau(\phi h, t_{h-1\to\psi\phi h}(b))} {}_{\phi h, t_{h-1\to\psi\phi h}(b)}^{\qquad q}\tau(h-1, b)$$

$$\to \bigsqcup_{q\in{}_\psi\tau(\phi h, t_{h\to\psi\phi h}(t_h(b)))} {}_{\phi h, t_{h\to\psi\phi h}(t_h(b))}^{\qquad q}\tau(h, t_h(b)) = \tau(h, t_h(b)).$$

Assume now that $\phi(h-1) < \phi(h)$. Equivalence (0.12) written as

$$x \leqslant \psi(y) \Longleftrightarrow \phi(x) \leqslant y$$

for any $x \in [I]$, $y \in [J]$, implies the counit $\phi\psi y \leqslant y$, the unit $x \leqslant \psi\phi x$, the identities $\psi\phi\psi = \psi$ and $\phi\psi\phi = \phi$. Its equivalent form

$$x > \psi(y) \Longleftrightarrow \phi(x) > y$$

implies in our case inequalities $h - 1 \leqslant \psi\phi(h-1) \leqslant \psi(\phi h - 1) < h \leqslant \psi\phi h$. Hence, $h - 1 = \psi\phi(h-1) = \psi(\phi h - 1)$.

If $y < g \in [J]$ and $\psi y = \psi g$, then for each collection $({}_\psi\tau, {}_{g,c}^{\ q}\tau)$ and for any $c \in t_\psi(y) = t_\psi(g)$ we have

$${}_\psi\tau_{(y,c)\to(g,c)} = \mathrm{id} : {}_\psi\tau(y, c) \to {}_\psi\tau(g, c).$$

In fact, it suffices to see it for $y = g - 1$. The preimage of any $q \in {}_\psi\tau(g, c)$ consists of one point, since ${}_{g,c}^{\ q}\tau(\psi(g-1), c) = {}_{g,c}^{\ q}\tau(\psi g, c) = \mathbf{1}$. Hence, ${}_\psi\tau_{(g-1,c)\to(g,c)}$ is an order preserving bijection.

For $b \in t(h-1)$ define $c = t_{h\to\psi\phi h}(t_h(b)) = t_{h-1\to\psi\phi h}(b)$. Let us define the successor map in the second case:

$$\tau(h-1, b) = \bigsqcup_{q\in{}_\psi\tau(\phi(h-1),b)} {}_{\phi(h-1),b}^{\qquad\ q}\tau(h-1, b) \to \bigsqcup_{p\in{}_\psi\tau(\phi h,c)} {}_{\phi h,c}^{\ \ p}\tau(h, t_h(b)) = \tau(h, t_h(b)).$$

It will map summand to summand in accordance with the mapping of indexing sets chosen as

$$_\psi\tau_{(\phi(h-1),b)\to(\phi h,c)} : {_\psi\tau}(\phi(h-1),b) \to {_\psi\tau}(\phi h,c), \qquad q \mapsto p.$$

We have chosen an arbitrary q and took its image p. In order to specify a mapping $\gamma : {_{\phi(h-1),b}^{\;q}}\tau(h-1,b) \to {_{\phi h,c}^{\;p}}\tau(h,t_h(b))$ we notice that the source is $\mathbf{1}$ embedded into ${_\psi\tau}(\phi(h-1),b) = {_\psi\tau}(\phi h-1,b)$ via $1 \mapsto q$. By definition of p

$$q \in {_\psi\tau}^{-1}_{(\phi h-1,b)\to(\phi h,c)}(p) \simeq {_{\phi h,c}^{\;p}}\tau(\psi(\phi h-1),b) = {_{\phi h,c}^{\;p}}\tau(h-1,b)$$

and q is represented by an element q' of the latter set. Finally, $\gamma(1) \overset{\text{def}}{=} {_{\phi h,c}^{\;p}}\tau_h(q')$.

In the particular case of $h = \psi g$ we take into account that $\psi\phi\psi g = \psi g$ and this formula gives

$$\tau(\psi g,b) = \bigsqcup_{q\in{_\psi\tau}(\phi\psi g,b)} {_{\phi\psi g,b}^{\;q}}\tau(\psi g,b) = \bigsqcup_{q\in{_\psi\tau}(\phi\psi g,b)} \mathbf{1} = {_\psi\tau}(\phi\psi g,b) = {_\psi\tau}(g,b),$$

as it should be. Further verifications show that the disassembling and assembling maps for t-trees are indeed inverse to each other, and invertibility of λ^f follows.

1.10 Example. The Monoidal product in the category $\mathsf{DG}(1) = \mathbf{dg}^{\mathbb{N}}$ of collections $\mathcal{A}_h$ is isomorphic to $\circledast(lt_I)(\mathcal{A}_h)_{h\in I}(z)$:

$$(\odot^{h\in I}\mathcal{A}_h)(z) = \bigoplus_{|\tau(0)|=z}^{\text{tree }\tau:[I]\to\mathcal{O}_{\mathsf{sk}}} \bigotimes^{h\in I} \bigotimes^{p\in\tau(h)} \mathcal{A}_h(|\tau_h^{-1}p|).$$

This turns $\mathbf{dg}^{\mathbb{N}}$ into a quite familiar Monoidal category. Algebras in this Monoidal category are precisely $\mathbf{dg}$-operads.

1.11 Definition. A *lax $\mathcal{C}at$-span multifunctor* $(F,\phi^I) : (\mathsf{L},\circledast^I_{\mathsf{L}},\lambda^f_{\mathsf{L}},\rho_{\mathsf{L}}) \to (\mathsf{M},\circledast^I_{\mathsf{M}},\lambda^f_{\mathsf{M}},\rho_{\mathsf{M}})$ between lax $\mathcal{C}at$-span multicategories is

i) a 1-morphism $F = ({_\mathsf{t}F},F,{_\mathsf{t}F}) : \mathsf{L} = ({_\mathsf{t}\mathsf{L}},\mathsf{L},{_\mathsf{t}\mathsf{L}}) \to ({_\mathsf{t}\mathsf{M}},\mathsf{M},{_\mathsf{t}\mathsf{M}}) = \mathsf{M}$ in $\mathcal{SMQ}_{\mathcal{C}at}$;

ii) a 2-morphism for each set $I \in \operatorname{Ob}\mathcal{O}_{\mathsf{sk}}$

$$\begin{tikzcd} \boxdot^I\mathsf{L} \arrow[r, "\boxdot^I F"] \arrow[d, "\circledast^I_{\mathsf{L}}"'] & \boxdot^I\mathsf{M} \arrow[d, "\circledast^I_{\mathsf{M}}"] \arrow[dl, Rightarrow, "\phi^I"'] \\ \mathsf{L} \arrow[r, "F"] & \mathsf{M} \end{tikzcd} \tag{1.14}$$

such that

$$\begin{tikzcd} \boxdot^{\mathbf{1}}\mathsf{L} \arrow[r, "\boxdot^{\mathbf{1}}F"] \arrow[d, "\mathrm{P}"'] \arrow[d, shift left=8, "\circledast^{\mathbf{1}}_{\mathsf{L}}"] \arrow[d, phantom, shift left=4, "\overset{\rho_{\mathsf{L}}}{\Leftarrow}"] & \boxdot^{\mathbf{1}}\mathsf{M} \arrow[d, "\circledast^{\mathbf{1}}_{\mathsf{M}}"] \arrow[dl, Rightarrow, "\phi^{\mathbf{1}}"'] \\ \mathsf{L} \arrow[r, "F"] & \mathsf{M} \end{tikzcd} = \begin{tikzcd} \boxdot^{\mathbf{1}}\mathsf{L} \arrow[r, "\boxdot^{\mathbf{1}}F"] \arrow[d, "\mathrm{P}"'] \arrow[dr, phantom, "="] & \boxdot^{\mathbf{1}}\mathsf{M} \arrow[d, "\mathrm{P}"'] \arrow[d, shift left=8, "\circledast^{\mathbf{1}}_{\mathsf{M}}"] \arrow[d, phantom, shift left=4, "\overset{\rho_{\mathsf{M}}}{\Leftarrow}"] \\ \mathsf{L} \arrow[r, "F"] & \mathsf{M} \end{tikzcd}$$

and for every map $f : I \to J$ of $\mathcal{O}_{\mathsf{sk}}$ the following equation holds:

$$
\begin{tikzcd}
 & \boxdot^I \mathsf{L} \arrow[rr, "\boxdot^I F"] \arrow[dl, "\circledast^f_{\mathsf{L}}"'] & & \boxdot^I \mathsf{M} \arrow[dl, "\circledast^f_{\mathsf{M}}"] \arrow[dd, "\circledast^I_{\mathsf{M}}"] \arrow[dlll, Rightarrow, "\boxdot^{j\in J}\phi^{f^{-1}j}"'] \\
\boxdot^J \mathsf{L} \arrow[rr, "\boxdot^J F"'] \arrow[dr, "\circledast^J_{\mathsf{L}}"'] & & \boxdot^J \mathsf{M} \arrow[dl, Rightarrow, "\phi^J"'] \arrow[dr, "\circledast^J_{\mathsf{M}}"'] \arrow[r, phantom, "\Leftarrow \lambda^f_{\mathsf{M}}"] & {} \\
 & \mathsf{L} \arrow[rr, "F"] & & \mathsf{M}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \boxdot^I \mathsf{L} \arrow[r, "\boxdot^I F"] \arrow[dl, "\circledast^f_{\mathsf{L}}"'] \arrow[dd, "\circledast^I_{\mathsf{L}}"] & \boxdot^I \mathsf{M} \arrow[ddl, Rightarrow, "\phi^I"'] \arrow[dd, "\circledast^I_{\mathsf{M}}"] \\
\boxdot^J \mathsf{L} \arrow[dr, "\circledast^J_{\mathsf{L}}"'] \arrow[r, phantom, "\Leftarrow \lambda^f_{\mathsf{L}}"] & {} & \\
 & \mathsf{L} \arrow[r, "F"] & \mathsf{M}
\end{tikzcd}
$$

Here 2-morphism $\boxdot^{j\in J}\phi^{f^{-1}j}$ means the pasting

$$
\begin{tikzcd}
\boxdot^I \mathsf{L} \arrow[r, "\boxdot^I F"] \arrow[d, "\Lambda^f"'] \arrow[dr, phantom, "="] & \boxdot^I \mathsf{M} \arrow[d, "\Lambda^f"] \\
\boxdot^{j\in J} \boxdot^{f^{-1}j} \mathsf{L} \arrow[r, "\boxdot^{j\in J}\boxdot^{f^{-1}j}F"] \arrow[d, "\boxdot^{j\in J}\circledast^{f^{-1}j}_{\mathsf{L}}"'] & \boxdot^{j\in J} \boxdot^{f^{-1}j} \mathsf{M} \arrow[d, "\boxdot^{j\in J}\circledast^{f^{-1}j}_{\mathsf{M}}"] \arrow[dl, Rightarrow, "\boxdot^{j\in J}\phi^{f^{-1}j}"'] \\
\boxdot^J \mathsf{L} \arrow[r, "\boxdot^J F"] & \boxdot^J \mathsf{M}
\end{tikzcd}
$$

1.12 Proposition. *Any lax $\mathcal{C}at$-span multifunctor*

$$(F, \phi^I) : (\mathsf{L}, \circledast^I_{\mathsf{L}}, \lambda^f_{\mathsf{L}}, \rho_{\mathsf{L}}) \to (\mathsf{M}, \circledast^I_{\mathsf{M}}, \lambda^f_{\mathsf{M}}, \rho_{\mathsf{M}})$$

between lax $\mathcal{C}at$-span operads induces a lax Monoidal functor

$$(F(1), \phi^I(1)|) : (\mathsf{L}(1), \odot^I_{\mathsf{L}(1)}, \lambda^f_{\mathsf{L}(1)}, \rho_{\mathsf{L}(1)}) \to (\mathsf{M}(1), \odot^I_{\mathsf{M}(1)}, \lambda^f_{\mathsf{M}(1)}, \rho_{\mathsf{M}(1)}).$$

Proof. Let us restrict the given transformation

$$
\begin{tikzcd}
\boxdot^I \mathsf{L} \arrow[r, "\boxdot^I F"] \arrow[d, "\circledast^I_{\mathsf{L}}"'] & \boxdot^I \mathsf{M} \arrow[d, "\circledast^I_{\mathsf{M}}"] \arrow[dl, Rightarrow, "\phi^I"'] \\
\mathsf{L} \arrow[r, "F"] & \mathsf{M}
\end{tikzcd}
$$

to the component indexed by the linear tree lt_I. This yields a natural transformation

$$
\begin{tikzcd}
\mathsf{L}(1)^I \arrow[r, "F(1)^I"] \arrow[d, "\odot^I_{\mathsf{L}(1)}"'] & \mathsf{M}(1)^I \arrow[d, "\odot^I_{\mathsf{M}(1)}"] \arrow[dl, Rightarrow, "\phi^I(1)|"'] \\
\mathsf{L}(1) \arrow[r, "F(1)"] & \mathsf{M}(1)
\end{tikzcd}
$$

Two equations for ϕ^I from Definition 1.11 imply two equations for ψ^I. $\square$

1.13. $\mathcal{C}at$-span multinatural transformations.

1.14 Definition. A *$\mathcal{C}at$-span multinatural transformation* $\xi : (F, \phi^I) \to (G, \psi^I) : (\mathsf{L}, \circledast^I_{\mathsf{L}}, \lambda^f_{\mathsf{L}}, \rho_{\mathsf{L}}) \to (\mathsf{M}, \circledast^I_{\mathsf{M}}, \lambda^f_{\mathsf{M}}, \rho_{\mathsf{M}})$ between lax $\mathcal{C}at$-span multifunctors is a 2-morphism $({}_{\mathsf{t}}\xi, \xi, {}_{\mathsf{t}}\xi) : ({}_{\mathsf{t}}F, F, {}_{\mathsf{t}}F) \to ({}_{\mathsf{t}}G, G, {}_{\mathsf{t}}G) : ({}_{\mathsf{t}}\mathsf{L}, \mathsf{L}, {}_{\mathsf{t}}\mathsf{L}) \to ({}_{\mathsf{t}}\mathsf{M}, \mathsf{M}, {}_{\mathsf{t}}\mathsf{M})$ of $\mathcal{SMQ}_{\mathcal{C}at}$ such that for all $I \in \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}$

$$
\begin{tikzcd}
(\boxdot^I \mathsf{L}, {}_{\mathsf{t}}\mathsf{L}) \arrow[r, shift left=2, "(\boxdot^I F, {}_{\mathsf{t}}F)"] \arrow[r, shift right=2, "(\boxdot^I G, {}_{\mathsf{t}}G)"'] \arrow[r, phantom, "\Downarrow(\boxdot^I \xi, {}_{\mathsf{t}}\xi)"] \arrow[d, "(\circledast^I_{\mathsf{L}}, \mathrm{Id})"'] & (\boxdot^I \mathsf{M}, {}_{\mathsf{t}}\mathsf{M}) \arrow[d, "(\circledast^I_{\mathsf{M}}, \mathrm{Id})"] \arrow[dl, Rightarrow, "(\psi^I, \mathrm{id})"'] \\
(\mathsf{L}, {}_{\mathsf{t}}\mathsf{L}) \arrow[r, "(G, {}_{\mathsf{t}}G)"'] & (\mathsf{M}, {}_{\mathsf{t}}\mathsf{M})
\end{tikzcd}
=
\begin{tikzcd}
(\boxdot^I \mathsf{L}, {}_{\mathsf{t}}\mathsf{L}) \arrow[r, "(\boxdot^I F, {}_{\mathsf{t}}F)"] \arrow[d, "(\circledast^I_{\mathsf{L}}, \mathrm{Id})"'] & (\boxdot^I \mathsf{M}, {}_{\mathsf{t}}\mathsf{M}) \arrow[d, "(\circledast^I_{\mathsf{M}}, \mathrm{Id})"] \arrow[dl, Rightarrow, "(\phi^I, \mathrm{id})"] \\
(\mathsf{L}, {}_{\mathsf{t}}\mathsf{L}) \arrow[r, shift left=2, "(F, {}_{\mathsf{t}}F)"] \arrow[r, shift right=2, "(G, {}_{\mathsf{t}}G)"'] \arrow[r, phantom, "\Downarrow(\xi, {}_{\mathsf{t}}\xi)"] & (\mathsf{M}, {}_{\mathsf{t}}\mathsf{M})
\end{tikzcd}
. \tag{1.15}
$$

1.15 Proposition. *The collection $\mathcal{SMC}_{\mathcal{C}at}$ of lax $\mathcal{C}at$-span multicategories, lax $\mathcal{C}at$-span multifunctors and $\mathcal{C}at$-span multinatural transformations is a 2-category.*

Proof. Composition of lax $\mathcal{C}at$-span multifunctors and identity $\mathcal{C}at$-span multifunctors are the obvious ones. The horizontal and the vertical compositions of 2-morphisms are implied by the underlying mapping $\mathcal{SMC}_{\mathcal{C}at} \to \mathcal{SMQ}_{\mathcal{C}at}$, which is a functor at the level of objects and 1-morphisms and is injective on 2-morphisms. One only has to verify that the identity 2-morphisms are in $\mathcal{SMC}_{\mathcal{C}at}$ and that if the horizontal or the vertical composition of two 2-morphisms in $\mathcal{SMC}_{\mathcal{C}at}$ makes sense, then its value in $\mathcal{SMQ}_{\mathcal{C}at}$ belongs actually to $\mathcal{SMC}_{\mathcal{C}at}$. This is clear from the shape of equation (1.15). □

1.16 Proposition. *Let L, M be lax $\mathcal{C}at$-span operads. Any $\mathcal{C}at$-span multinatural transformation*

$$\xi : (F, \phi^I) \to (G, \psi^I) : (\mathsf{L}, \circledast^I_{\mathsf{L}}, \lambda^f_{\mathsf{L}}, \rho_{\mathsf{L}}) \to (\mathsf{M}, \circledast^I_{\mathsf{M}}, \lambda^f_{\mathsf{M}}, \rho_{\mathsf{M}})$$

between lax $\mathcal{C}at$-span multifunctors induces a Monoidal transformation

$$\xi(1) : (F(1), \phi^I(1)) \to (G(1), \psi^I(1)) : (\mathsf{L}(1), \odot^I_{\mathsf{L}(1)}, \lambda^f_{\mathsf{L}(1)}, \rho_{\mathsf{L}(1)}) \to (\mathsf{M}(1), \odot^I_{\mathsf{M}(1)}, \lambda^f_{\mathsf{M}(1)}, \rho_{\mathsf{M}(1)}).$$

Proof. Restricting the transformations to linear trees we get

$$
\begin{tikzcd}
\mathsf{L}(1)^I \arrow[r, shift left=2, "F(1)^I"] \arrow[r, shift right=2, "G(1)^I"'] \arrow[r, phantom, "\Downarrow\xi(1)^I"] \arrow[d, "\odot^I"'] & \mathsf{M}(1)^I \arrow[d, "\odot^I"] \arrow[dl, Rightarrow, "\psi^I(1)"'] \\
\mathsf{L}(1) \arrow[r, "G(1)"'] & \mathsf{M}(1)
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{L}(1)^I \arrow[r, "F(1)^I"] \arrow[d, "\odot^I"'] & \mathsf{M}(1)^I \arrow[d, "\odot^I"] \arrow[dl, Rightarrow, "\phi^I(1)"] \\
\mathsf{L}(1) \arrow[r, shift left=2, "F(1)"] \arrow[r, shift right=2, "G(1)"'] \arrow[r, phantom, "\Downarrow\xi(1)"] & \mathsf{M}(1)
\end{tikzcd}
. \tag{1.16}
$$

These equations say that the transformation $\xi(1)$ is Monoidal, see [BLM08, Definition 2.20]. □

1.17. dg-operads are lax $\mathcal{C}at$-span multifunctors $1 \to \mathsf{DG}$. Let C be an arbitrary lax $\mathcal{C}at$-span operad. Denote by $\mathsf{1}$ the unit object of the Monoidal category $({}^{\mathbb{1}}\mathcal{SMQ}_{\mathcal{C}at}, \boxdot^I, \Lambda^f, \mathrm{P})$ of lax $\mathcal{C}at$-span operads. This is a lax $\mathcal{C}at$-span operad with

$$\mathsf{1}(n) = \begin{cases} \mathbb{1} = \text{terminal (1-morphism) category}, & \text{if } n = 1, \\ \varnothing = \text{initial (empty) category}, & \text{if } n \neq 1. \end{cases}$$

A multiquiver morphism $\mathsf{1} \to \mathsf{C}$ is a functor $\mathbb{1} \to \mathsf{C}(1)$, so it is just an object of $\mathsf{C}(1)$. In particular, a $\mathcal{C}at$-multiquiver map $\mathsf{1} \to \mathsf{DG}$ is the same as a functor $\mathbb{1} \to \mathsf{DG}(1) = \mathbf{dg}^{\mathbb{N}}$.

Proposition 1.12 implies that a lax $\mathcal{C}at$-span multifunctor $\mathsf{1} \to \mathsf{DG}$ is the same as a Monoidal functor $\mathbb{1} \to \mathsf{DG}(1) = (\mathbf{dg}^{\mathbb{N}}, \odot^I)$. By Definition 2.25 and Proposition 2.28 of [BLM08] this is the same as an algebra in $(\mathbf{dg}^{\mathbb{N}}, \odot^I)$, that is, an operad.

By Proposition 1.16 a multinatural transformation $\xi : (F, \phi^I) \to (G, \psi^I) : \mathsf{1} \to \mathsf{DG}$ is the same as a Monoidal transformation $\xi(1) : \mathcal{O} = (F(1), \phi^I(1)) \to (G(1), \psi^I(1)) = \mathcal{P} : \mathbb{1} \to (\mathbf{dg}^{\mathbb{N}}, \odot^I)$. Here operads $\mathcal{O}$ and $\mathcal{P}$ are identified with the image of $1 \in \operatorname{Ob}\mathbb{1}$ under corresponding functors. Equations (1.16) for $\xi(1)$ translate to

$$\big(\odot^I \mathcal{O} \xrightarrow{\odot^I \xi(1)} \odot^I \mathcal{P} \xrightarrow{\psi^I(1)} \mathcal{P}\big) = \big(\odot^I \mathcal{O} \xrightarrow{\phi^I(1)} \mathcal{O} \xrightarrow{\xi(1)} \mathcal{P}\big),$$

that is, $\xi(1) : \mathcal{O} \to \mathcal{P}$ is a morphism of operads.

1.18. $n \wedge 1$-operad modules are lax $\mathcal{C}at$-span multifunctors. Consider a $\mathcal{C}at$-multicategory L_n with $\operatorname{Ob}\mathsf{L}_n = \{0, 1, 2, \dots, n\}$ such that

$$\begin{aligned} \mathsf{L}_n(i; i) &= \mathbb{1} && \text{for } 0 \leqslant i \leqslant n, \\ \mathsf{L}_n(1, 2, \dots, n; 0) &= \mathbb{1} && \text{and} \\ \mathsf{L}_n(k_1, k_2, \dots, k_m; k_0) &= \varnothing && \text{for other lists of arguments.} \end{aligned}$$

Thus, L_n is a discrete category with the list of objects $(0;0)$, $(1;1)$, $\dots$, $(n;n)$ and $(1, 2, \dots, n; 0)$. Components of the category $\boxdot^{\mathbf{k}}\mathsf{L}_n$ either are empty or indexed by trees of two kinds:

- labelled linear trees $(lt_{\mathbf{k}}, {}^k i) = (i \to i \to \cdots \to i)$, whose all vertices are labelled with the same $i \in [n]$;
- labelled trees with $1+l$ vertices labelled with 1 (or 2, or n) and $k-l$ vertices labelled with 0

$$(t, \ell) = \succ^k_l = \begin{array}{l} n \longrightarrow \cdots \longrightarrow n \longrightarrow n \\ \cdots \longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots \\ 2 \longrightarrow \cdots \longrightarrow 2 \longrightarrow 2 \\ 1 \longrightarrow \cdots \longrightarrow 1 \longrightarrow 1 \end{array} \rightarrow 0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

The non-vanishing components are terminal categories $\mathbb{1}$. The 1-morphism $\circledast^{\mathbf{k}} : \boxdot^{\mathbf{k}}\mathsf{L}_n \to \mathsf{L}_n$ is $\mathrm{Id}_{\mathbb{1}}$ on any non-vanishing component of the source. It takes the component indexed by a tree of the first kind to the full subcategory $\{(i;i)\} \subset \mathsf{L}_n$. The component indexed by a tree of the second kind goes to the full subcategory $\{(1,2,\dots,n;0)\} \subset \mathsf{L}_n$. The transformation λ^f and ρ are $\mathrm{id}_{\mathrm{Id}_{\mathbb{1}}}$ on non-vanishing components.

1.19 Definition. An $n \wedge 1$-*operad module* is a lax $\mathcal{C}at$-span multifunctor $\mathcal{P} : \mathsf{L}_n \to \mathsf{DG}$. A *morphism of* $n \wedge 1$-*operad modules* $r : \mathcal{P} \to \mathcal{Q}$ is a $\mathcal{C}at$-span multinatural transformation $r : \mathcal{P} \to \mathcal{Q} : \mathsf{L}_n \to \mathsf{DG}$. The disjoint union M over $n \geqslant 0$ of so defined categories ${}_n\mathrm{Op}_1$ of $n \wedge 1$-operad modules is the category of operad polymodules.

Let us describe the structure of an $n \wedge 1$-operad module. A multiquiver 1-morphism $\mathsf{L}_n \to \mathsf{DG}$ amounts to a sequence $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$, where $\mathcal{B},\mathcal{A}_i \in \mathrm{Ob}\,\mathbf{dg}^{\mathbb{N}}$ for $i \in \mathbf{n}$, and $\mathcal{P} \in \mathrm{Ob}\,\mathbf{dg}^{\mathbb{N}^n}$.

1.20 Example. Particular cases of $\circledast$ for DG will obtain a special notation. In addition to the above assume that $\mathcal{A}_i^h \in \mathrm{Ob}\,\mathbf{dg}^{\mathbb{N}}$. We denote

$$\begin{gathered}
\mathcal{P} \odot_0 \mathcal{B} = \circledast(\mathbf{n} \to \mathbf{1} \to \mathbf{1})(\mathcal{P};\mathcal{B}),\\
\odot_{>0}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{P}) = \circledast(\mathbf{n} \xrightarrow{1} \mathbf{n} \to \mathbf{1})((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{P}),\\
\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{P};\mathcal{B}) = \circledast(\mathbf{n} \xrightarrow{1} \mathbf{n} \to \mathbf{1} \to \mathbf{1})((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{P};\mathcal{B}).
\end{gathered}$$

All these expressions describe actions of several copies of the category $\mathbf{dg}^{\mathbb{N}}$ on the category $\mathbf{dg}^{\mathbb{N}^n}$. In isomorphic form these actions are given by the graded components, $\ell \in \mathbb{N}^n$,

$$(\mathcal{P} \odot_0 \mathcal{B})(\ell) \simeq \bigoplus_{t_1+\dots+t_m=\ell}^{m\geqslant 0} \Bigl(\bigotimes_{r=1}^{m} \mathcal{P}(t_r)\Bigr) \otimes \mathcal{B}(m),$$

$$\odot_{>0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P})(\ell) \simeq \bigoplus_{k\in\mathbb{N}^n} \bigoplus_{j_1^i+\dots+j_{k^i}^i=\ell^i}^{\forall\, i\in\mathbf{n}} \Bigl[\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p^i)\Bigr] \otimes \mathcal{P}(k),$$

$$\begin{aligned}
&\odot_{\geqslant 0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})(\ell)\\
&\simeq \bigoplus_{m=0}^{\infty} \bigoplus_{k_1,\dots,k_m\in\mathbb{N}^n} \bigoplus_{\sum_{p=1}^{k_1^i+\dots+k_m^i} j_p^i=\ell^i}^{\forall\, i\in\mathbf{n}} \Bigl(\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k_1^i+\dots+k_m^i} \mathcal{A}_i(j_p^i)\Bigr) \otimes \Bigl(\bigotimes_{r=1}^{m} \mathcal{P}(k_r)\Bigr) \otimes \mathcal{B}(m)\\
&\simeq \bigoplus_{m=0}^{\infty} \bigoplus_{t_1+\dots+t_m=\ell} \bigoplus_{k_1,\dots,k_m\in\mathbb{N}^n} \bigoplus_{y_{r,1}^i+\dots+y_{r,k_r^i}^i=t_r^i}^{\forall\, i\in\mathbf{n},\, r\in\mathbf{m}} \Bigl[\bigotimes_{r=1}^{m} \Bigl(\bigotimes_{i=1}^{n} \bigotimes_{v=1}^{k_r^i} \mathcal{A}_i(y_{r,v}^i)\Bigr) \otimes \mathcal{P}(k_r)\Bigr] \otimes \mathcal{B}(m).
\end{aligned}$$

Actually, the action $\odot_{>0}$ can be presented as a combination of partial actions $\odot_i$ for $1 \leqslant i \leqslant n$ defined as

$$\mathcal{A} \odot_i \mathcal{P} = \odot_{>0}(\mathbb{1},\dots,\mathbb{1},\mathcal{A},\mathbb{1},\dots,\mathbb{1};\mathcal{P}), \qquad \mathcal{A} \text{ on } i\text{-th place}.$$

Explicit presentation of this action is

$$(\mathcal{A} \odot_i \mathcal{P})(\ell) = \bigoplus_{j_1+\dots+j_q=\ell^i}^{q\geqslant 0} \Big(\bigotimes_{p=1}^{q} \mathcal{A}(j_p)\Big) \otimes \mathcal{P}(\ell, \ell^i \mapsto q),$$

where $(\ell, \ell^i \mapsto q) = (\ell^1, \dots, \ell^{i-1}, q, \ell^{i+1}, \dots, \ell^n)$.

Iterating these actions I times we get the following expressions

$$\odot_0^{[I]}(\mathcal{P}; (\mathcal{B}_i)_{i\in I}) = \circledast(\mathbf{n} \to \underbrace{\mathbf{1} \to \mathbf{1} \cdots \to \mathbf{1}}_{\mathbf{1}\sqcup I})(\mathcal{P}; (\mathcal{B}_i)_{i\in I}),$$

$$\odot_{>0}^{I\sqcup\mathbf{1}}(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I}; \mathcal{P}) = \circledast(\underbrace{\mathbf{n} \xrightarrow{1} \mathbf{n} \xrightarrow{1} \mathbf{n} \dots \xrightarrow{1} \mathbf{n} \xrightarrow{1} \mathbf{n}}_{\mathbf{1}\sqcup I} \to \mathbf{1})(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I}; \mathcal{P}),$$

$$\odot_{\geqslant 0}^{[I]}(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I}; \mathcal{P}; (\mathcal{B}_i)_{i\in I}) = \circledast(\underbrace{\mathbf{n} \dots \xrightarrow{1} \mathbf{n}}_{[I]} \to \underbrace{\mathbf{1} \to \dots \mathbf{1}}_{[I]})(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I}; \mathcal{P}; (\mathcal{B}_i)_{i\in I}).$$

Here the last three trees are functors $\mathbf{1} \sqcup [I] \to \mathcal{O}_{\mathsf{sk}}$, $\mathbf{1} \sqcup [I]^{\mathrm{op}} \to \mathcal{O}_{\mathsf{sk}}$ and $\mathbf{1} \sqcup [I]^{\mathrm{op}} \cup_{0\sim 0} [I] \to \mathcal{O}_{\mathsf{sk}}$ respectively, where $[I]^{\mathrm{op}} \cup_{0\sim 0} [I]$ is obtained by identifying elements $0 \in [I]^{\mathrm{op}}$ and $0 \in [I]$.

One can show that these actions are Monoidal and that actions $\odot_i$ for different $0 \leqslant i \leqslant n$ commute up to isomorphisms that satisfy coherence conditions. Furthermore, the action $\odot_{\geqslant 0}$ can be presented as a combination of partial actions $\odot_i$ for $0 \leqslant i \leqslant n$. To be rigorous this approach requires more definitions, and we have chosen to avoid it.

Thus, a lax $\mathcal{C}at$-span multifunctor $(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) : \mathsf{L}_n \to \mathsf{DG}$ consists of a coherent system of action maps

$$\circledast(\underbrace{\mathbf{n} \xrightarrow{1} \dots \xrightarrow{1} \mathbf{n}}_{[I]} \to \underbrace{\mathbf{1} \to \dots \to \mathbf{1}}_{[J]})(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I}; \mathcal{P}; (\mathcal{B}_j)_{j\in J}) \to \mathcal{P}.$$

Since the actions are Monoidal, this is equivalent to giving a coherent system of actions for $J = I$ only, that is, a system of morphisms $\alpha^I : \odot_{\geqslant 0}^{[I]}(((\mathcal{A}_i^h)_{h\in\mathbf{n}})_{i\in I}; \mathcal{P}; (\mathcal{B}_i)_{i\in I}) \to \mathcal{P}$. Equivalently, we consider the monad $\mathcal{P} \mapsto \odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$.

1.21 Definition. An $n\wedge 1$-*operad module* can be defined also as a family $(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$, consisting of $n+1$ operads $\mathcal{A}_i$, $\mathcal{B}$ and an object $\mathcal{P} \in \mathbf{gr}^{\mathbb{N}^n}$ (resp. $\mathcal{P} \in \mathbf{dg}^{\mathbb{N}^n}$), equipped with an algebra structure

$$\alpha : \odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to \mathcal{P}$$

for the monad $\mathcal{Q} \mapsto \odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{Q}; \mathcal{B})$.

An action is specified by a collection of maps given for each $m \in \mathbb{N}$, each family $k_1, \dots, k_m \in \mathbb{N}^n$ and each family of non-negative integers $\big((j_p^i)_{p=1}^{k_1^i+\dots+k_m^i}\big)_{i=1}^n$

$$\alpha : \Big(\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k_1^i+\dots+k_m^i} \mathcal{A}_i(j_p^i)\Big) \otimes \Big(\bigotimes_{r=1}^{m} \mathcal{P}(k_r)\Big) \otimes \mathcal{B}(m) \to \mathcal{P}\Big(\Big(\sum_{p=1}^{k_1^i+\dots+k_m^i} j_p^i\Big)_{i=1}^{n}\Big). \tag{1.17}$$

Assume that $f_i : \mathcal{C}_i \to \mathcal{A}_i$, $g : \mathcal{D} \to \mathcal{B}$ are morphisms of operads. They imply a morphism of monads $\odot_{\geqslant 0}(\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{Q}; \mathcal{D}) \to \odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{Q}; \mathcal{B})$. An algebra $\mathcal{P}$ over the latter monad becomes an algebra over the former monad denoted ${}_{f_1,\dots,f_n}\mathcal{P}_g$.

Restricting the action α to submonads $\odot_{>0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{Q}) \hookrightarrow \odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{Q}; \mathcal{B})$, $\mathcal{Q} \odot_0 \mathcal{B} \hookrightarrow \odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{Q}; \mathcal{B})$, $\mathcal{A}_i \odot_i \mathcal{Q} \hookrightarrow \odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{Q}; \mathcal{B})$, $1 \leqslant i \leqslant n$, obtained via insertion of operad units η, we get the partial actions

$$
\begin{gathered}
\lambda = \lambda_{k,(j^i_p)} : \Bigl[\bigotimes_{i=1}^{n} \bigotimes_{p=1}^{k^i} \mathcal{A}_i(j^i_p)\Bigr] \otimes \mathcal{P}\bigl((k^i)_{i=1}^n\bigr) \to \mathcal{P}\Bigl(\Bigl(\sum_{p=1}^{k^i} j^i_p\Bigr)_{i=1}^{n}\Bigr), \\
\rho = \rho_{(k_r)} : \Bigl(\bigotimes_{r=1}^{m} \mathcal{P}(k_r)\Bigr) \otimes \mathcal{B}(m) \to \mathcal{P}\Bigl(\sum_{r=1}^{m} k_r\Bigr), \qquad (1.18) \\
\lambda^i = \lambda^i_{k,(j_p)} : \Bigl[\bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p)\Bigr] \otimes \mathcal{P}(k) \to \mathcal{P}\Bigl(k^1, \dots, k^{i-1}, \sum_{p=1}^{k^i} j_p, k^{i+1}, \dots, k^n\Bigr).
\end{gathered}
$$

On the other hand, given n left actions λ^i of $\mathcal{A}_i$ and a right action ρ of $\mathcal{B}$, all pairwise commuting, we can restore the total action α.

The category of $n \wedge 1$-operad modules ${}_n\mathrm{Op}_1$ has morphisms

$$
(f_1, \dots, f_n; h; f_0) : (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{A}_0) \to (\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{Q}; \mathcal{C}_0),
$$

where $f_i : \mathcal{A}_i \to \mathcal{C}_i$, $0 \leqslant i \leqslant n$, are morphisms of $\mathbf{dg}$-operads and $h : \mathcal{P} \to {}_{f_1,\dots,f_n}\mathcal{Q}_{f_0} \in \mathbf{dg}^{\mathbb{N}^n}$ is a module morphism with respect to actions of all $\mathcal{A}_i$. In fact, a morphism of $n \wedge 1$-operad modules is by definition a pair of natural transformations $\xi : \mathcal{P} = (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{A}_0) \to (\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{Q}; \mathcal{C}_0) = \mathcal{Q} : \mathsf{L}_n \to \mathsf{DG}$, $\mathrm{id} : \triangleright \to \triangleright : \{0, 1, \dots, n\} \to \mathbf{1}$ which satisfy certain equations. Since L_n is a discrete category with $n{+}2$ objects, the transformation ξ consists of morphisms $f_i : \mathcal{A}_i \to \mathcal{C}_i \in \mathbf{dg}^{\mathbb{N}}$, $0 \leqslant i \leqslant n$, and $h : \mathcal{P} \to \mathcal{Q} \in \mathbf{dg}^{\mathbb{N}^n}$. The category $\boxdot^{\mathbf{k}}\mathsf{L}_n$ is also discrete and its objects are $(lt_{\mathbf{k}}, {}^k i)$ and $\succ^k_l$, see Section 1.18. Equation (1.15) on $\boxdot^{\mathbf{k}}\mathsf{L}_n$ reads on objects $(lt_{\mathbf{k}}, {}^k i)$ as

$$
\bigl(\odot^{\mathbf{k}}\mathcal{A}_i \xrightarrow{\odot^{\mathbf{k}} f_i} \odot^{\mathbf{k}}\mathcal{B}_i \xrightarrow{\mu^k} \mathcal{B}_i\bigr) = \bigl(\odot^{\mathbf{k}}\mathcal{A}_i \xrightarrow{\mu^k} \mathcal{A}_i \xrightarrow{f_i} \mathcal{B}_i\bigr),
$$

that is, f_i is a morphism of operads. On the object $\succ^k_l$ the equation gives

$$
\begin{aligned}
\bigl[\circledast^{\mathbf{k}}_{\mathsf{DG}}(\succ^k_l)(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{A}_0) &\xrightarrow{\circledast^{\mathbf{k}}_{\mathsf{DG}}(\succ^k_l)(f_1,\dots,f_n;h;f_0)} \circledast^{\mathbf{k}}_{\mathsf{DG}}(\succ^k_l)(\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{Q}; \mathcal{C}_0) \xrightarrow{\alpha} \mathcal{Q}\bigr] \\
&= \bigl[\circledast^{\mathbf{k}}_{\mathsf{DG}}(\succ^k_l)(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{A}_0) \xrightarrow{\alpha} \mathcal{P} \xrightarrow{h} \mathcal{Q}\bigr],
\end{aligned}
$$

that is, $h : \mathcal{P} \to {}_{f_1,\dots,f_n}\mathcal{Q}_{f_0}$ is a module morphism with respect to actions of all $\mathcal{A}_i$, $0 \leqslant i \leqslant n$.

Objects of the category $\mathbf{dg}^{n\mathbb{N}\sqcup\mathbb{N}^n\sqcup\mathbb{N}}$ are also written as tuples $(\mathcal{U}_1, \dots, \mathcal{U}_n; \mathcal{X}; \mathcal{W})$. The free algebra functor for the monad $\odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \text{-}; \mathcal{B})$ is the functor $\mathbf{dg}^{\mathbb{N}^n} \to$

$\mathcal{A}_1$-$\cdots$-$\mathcal{A}_n$-mod-$\mathcal{B}$, $\mathcal{X} \mapsto \odot_{\geqslant 0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{X};\mathcal{B})$, left adjoint to the underlying functor $\mathcal{A}_1$-$\cdots$-$\mathcal{A}_n$-mod-$\mathcal{B} \to \mathbf{dg}^{\mathbb{N}^n}$. Hence, there is also a pair of adjoint functors $F : \mathbf{dg}^{n\mathbb{N}\sqcup\mathbb{N}^n\sqcup\mathbb{N}} \rightleftarrows {}_n\mathrm{Op}_1 : U$,

$$F(\mathcal{U}_1,\dots,\mathcal{U}_n;\mathcal{X};\mathcal{W}) = (T\mathcal{U}_1,\dots,T\mathcal{U}_n;\odot_{\geqslant 0}(T\mathcal{U}_1,\dots,T\mathcal{U}_n;\mathcal{X};T\mathcal{W});T\mathcal{W}).$$

The module part is indexed by trees with the top floor describing $\mathcal{X}(\text{-}_1)\otimes\cdots\otimes\mathcal{X}(\text{-}_k)$, lower floors indexed by $\mathcal{W}(\text{-})$ and n forests indexed by $\mathcal{U}_i(\text{-})$ attached to each of k leaves.

In particular,

$$F(\mathcal{U}_1,\dots,\mathcal{U}_n;0;\mathcal{W}) = (T\mathcal{U}_1,\dots,T\mathcal{U}_n;(T\mathcal{W})(0);T\mathcal{W}).$$

Recall that for any object $A = (\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ and any collection of complexes M the object $A\langle M,0\rangle$ is isomorphic to $F(M[1])\sqcup A$, see Section 0.8.

1.22. The monad of free $n\wedge 1$-operad modules. Recall [BW05, Section 3.3.6] that a parallel pair of morphisms $f,g : A\to B\in\mathcal{C}$ is called *reflexive* if there is a morphism $r : B\to A\in\mathcal{C}$ such that $f\circ r = \mathrm{id}_B = g\circ r$. Recall that a *contractible coequalizer* [BW05, Section 3.3.3] (= a *split fork* [Mac88, Section VI.6]) is a diagram in a category $\mathcal{D}$

$$\begin{tikzcd} A' \arrow[r, shift left=2, "d^0"] \arrow[r, shift right=2, "d^1"'] & B' \arrow[l, "t"' description] \arrow[r, shift left, "d"] & C' \arrow[l, shift left, "s"] \end{tikzcd}$$

such that $d^0\circ t = \mathrm{id}$, $d^1\circ t = s\circ d$, $d\circ s = \mathrm{id}$, and $d\circ d^0 = d\circ d^1$. Suppose there is a functor $U : \mathcal{C}\to\mathcal{D}$. Then a pair $f,g : A\to B\in\mathcal{C}$ is called *U-contractible coequalizer pair* if $d^0 = Uf, d^1 = Ug : UA\to UB$ extend to a contractible coequalizer in $\mathcal{D}$. One says that U *creates* U-contractible coequalizers [Mac88, Section VI.7] if for any pair $f,g : A\to B\in\mathcal{C}$ and any contractible coequalizer in $\mathcal{D}$

$$\begin{tikzcd} UA \arrow[r, shift left=2, "Uf"] \arrow[r, shift right=2, "Ug"'] & UB \arrow[l, "t"' description] \arrow[r, shift left, "d"] & C' \arrow[l, shift left, "s"] \end{tikzcd}$$

† there is a unique morphism $h : B\to C\in\mathcal{C}$ such that $C' = UC$, $d = Uh$, and

‡ h is a coequalizer of (f,g) in $\mathcal{C}$.

The following statement is Exercise 3.3.(PPTT) of [BW05].

1.23 Theorem. *Let $U : \mathcal{C}\to\mathcal{D}$ be a functor which has a left adjoint F. Then the comparison functor $\Phi : \mathcal{C}\to\mathcal{D}^\top$, $A\mapsto (UA, U\varepsilon : UFUA\to UA)$, for the monad $\top = U\circ F$ in $\mathcal{D}$ is an isomorphism of categories if and only if U creates coequalizers of reflexive U-contractible coequalizer pairs in $\mathcal{C}$.*

Here $\mathcal{D}^\top$ is the category of $\top$-algebras. The condition of the theorem applied to $f = \mathrm{id} = g$ implies that U reflects isomorphisms. The proof of this theorem is contained in the proof of (PTT), Beck's Precise Tripleability Theorem [BW05, Theorem 3.3.14]. We shall use the following corollary to Theorem 1.23.

One says that (*cf.* [BW05, Section 3.5])

(CTT′) $U : \mathcal{C} \to \mathcal{D}$ creates coequalizers for reflexive pairs (f, g) for which (Uf, Ug) has a coequalizer,

if for any reflexive pair $f, g : A \to B \in \mathcal{C}$ and any coequalizer $d : UB \to C'$ of (Uf, Ug) in $\mathcal{D}$ conclusions † and ‡ before Theorem 1.23 hold.

1.24 Corollary (Crude Tripleability Theorem)**.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor which satisfies* (CTT′) *and has a left adjoint F. Then the comparison functor $\Phi : \mathcal{C} \to \mathcal{D}^\top$, $\top = U \circ F$ is an isomorphism of categories.*

1.25 Definition. An *ideal* of an object $\tilde{\mathcal{A}} = (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{A}_0) \in {}_n\mathrm{Op}_1$ is a subobject $\tilde{\mathcal{I}} = (\mathcal{I}_1, \dots, \mathcal{I}_n; \mathcal{K}; \mathcal{I}_0)$ of $U\tilde{\mathcal{A}}$ in $\mathbf{dg}^S$, $S = n\mathbb{N} \sqcup \mathbb{N}^n \sqcup \mathbb{N} = \mathbb{N} \sqcup \cdots \sqcup \mathbb{N} \sqcup \mathbb{N}^n \sqcup \mathbb{N}$, stable under all multiplications in operads $\mathcal{A}_j$, $0 \leqslant j \leqslant n$, and under the action on $\mathcal{P}$ from (1.17). Namely if at least one $\otimes$-argument of multiplication or action is in $\tilde{\mathcal{I}}$, then the result is in $\tilde{\mathcal{I}}$ as well. Equivalently, for all values of indices

$$\lambda^i_{k,(j_p)}\Biggl(\Biggl[\bigotimes_{p=1}^{k^i} \mathcal{A}_i(j_p)\Biggr] \otimes \mathcal{K}(k)\Biggr) \subset \mathcal{K}\Biggl(k^1, \dots, k^{i-1}, \sum_{p=1}^{k^i} j_p, k^{i+1}, \dots, k^n\Biggr),$$

$$\rho\Biggl(\Biggl[\Bigl(\bigotimes_{r=1}^{t-1} \mathcal{P}(k_r)\Bigr) \otimes \mathcal{K}(k_t) \otimes \Bigl(\bigotimes_{r=t+1}^{m} \mathcal{P}(k_r)\Bigr)\Biggr] \otimes \mathcal{A}_0(m)\Biggr) \subset \mathcal{K}\Biggl(\sum_{r=1}^{m} k_r\Biggr),$$

and $\mathcal{I}_j$ are ideals of operads $\mathcal{A}_j$ in a similar sense.

Ideals are precisely kernels in $\mathbf{dg}^S$ of Uh for morphisms $h : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}} \in {}_n\mathrm{Op}_1$. If $\tilde{\mathcal{I}}$ is an ideal of $\tilde{\mathcal{A}}$, then the quotient $U\tilde{\mathcal{A}}/U\tilde{\mathcal{I}}$ in the abelian category $\mathbf{dg}^S$ admits a unique structure of an $n \wedge 1$-operad module such that the quotient map $q : \mathcal{A} \to \tilde{\mathcal{A}}/\tilde{\mathcal{I}}$ is in ${}_n\mathrm{Op}_1$.

For any subcomplex $\mathcal{N} \subset \tilde{\mathcal{A}} \in \mathbf{dg}^S$ there is the smallest ideal $\tilde{\mathcal{I}}$ of $\tilde{\mathcal{A}}$ containing $\mathcal{N}$. It is spanned as a graded $\Bbbk$-submodule of $\tilde{\mathcal{A}}$ by results of multiplications or actions containing an element of $\mathcal{N}$ among its $\otimes$-arguments. So obtained $\tilde{\mathcal{I}}$ is indeed an ideal due to associativity of the action. In particular, for a pair of parallel arrows $f, g : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}} \in {}_n\mathrm{Op}_1$ there is the image $\mathcal{N} = \mathrm{Im}(f - g)$ in the abelian category $\mathbf{dg}^S$. If $\tilde{\mathcal{I}}$ is the smallest ideal of $\tilde{\mathcal{A}}$ containing $\mathcal{N}$, then the quotient $\tilde{\mathcal{A}}/\tilde{\mathcal{I}}$ is the coequalizer of f and g in ${}_n\mathrm{Op}_1$.

1.26 Proposition. *The comparison functor for the underlying functor $U : {}_n\mathrm{Op}_1 \to \mathbf{dg}^{n\mathbb{N} \sqcup \mathbb{N}^n \sqcup \mathbb{N}}$ is an isomorphism of categories.*

Thus ${}_n\mathrm{Op}_1$ is isomorphic to the category of $\top$-algebras for the monad $\top = U \circ F$ in $\mathbf{dg}^{n\mathbb{N} \sqcup \mathbb{N}^n \sqcup \mathbb{N}}$.

Proof. Let us prove that U satisfies condition (CTT′). First (as a warm-up) we show it for $U : \mathrm{Op} \to \mathbf{dg}^{\mathbb{N}}$. Let a reflexive pair $f, g : \mathcal{A} \leftrightarrows \mathcal{B} : r$ in Op be given together with a coequalizer $d : U\mathcal{B} \to \mathcal{C}'$ in $\mathbf{dg}^{\mathbb{N}}$ of (Uf, Ug). The subobject $\mathcal{K} = \mathrm{Im}(f-g) = \mathrm{Ker}\, d \in \mathbf{dg}^{\mathbb{N}}$ of $U\mathcal{B}$ is an ideal of the operad $\mathcal{B}$. In fact, for any multiplication μ from (0.2) for the operads $\mathcal{B}$ and $\mathcal{A}$ we have

$$\begin{aligned}
&\mu^{\mathcal{B}}(b_1 \otimes \cdots \otimes b_{i-1} \otimes (f-g)a \otimes b_{i+1} \otimes \cdots \otimes b_k \otimes b) \\
&= \mu^{\mathcal{B}}(frb_1 \otimes \cdots \otimes frb_{i-1} \otimes fa \otimes frb_{i+1} \otimes \cdots \otimes frb_k \otimes frb) \\
&- \mu^{\mathcal{B}}(grb_1 \otimes \cdots \otimes grb_{i-1} \otimes ga \otimes grb_{i+1} \otimes \cdots \otimes grb_k \otimes grb) \\
&= (f-g)\mu^{\mathcal{A}}(rb_1 \otimes \cdots \otimes rb_{i-1} \otimes a \otimes rb_{i+1} \otimes \cdots \otimes rb_k \otimes rb) \in \mathcal{K}
\end{aligned}$$

for all $1 \leqslant i \leqslant k$. Similarly,

$$\mu^{\mathcal{B}}(b_1 \otimes \cdots \otimes b_k \otimes (f-g)a) = (f-g)\mu^{\mathcal{A}}(rb_1 \otimes \cdots \otimes rb_k \otimes a) \in \mathcal{K}.$$

Thus the quotient operad $\mathcal{B}/\mathcal{K}$ exists together with the quotient map $q : \mathcal{B} \to \mathcal{B}/\mathcal{K} \in \mathrm{Op}$. The map d factorises as $d = \bigl(U\mathcal{B} \xrightarrow{Uq} U(\mathcal{B}/\mathcal{K}) \xrightarrow[\sim]{\phi} \mathcal{C}'\bigr)$ for a unique isomorphism $\phi \in \mathbf{dg}^{\mathbb{N}}$. Transferring the operad structure from $\mathcal{B}/\mathcal{K}$ to $\mathcal{C}'$ along ϕ we make the latter into an operad $\mathcal{C}$, make d into a morphism of operads. Clearly, properties † and ‡ on page 39 hold true.

Consider now a reflexive pair in ${}_n\mathrm{Op}_1$

$$\tilde{f} = ((f_i)_0^n; f), \tilde{g} = ((g_i)_0^n; g) : \tilde{\mathcal{A}} = ((\mathcal{A}_i)_1^n; \mathcal{P}; \mathcal{A}_0) \leftrightarrows \tilde{\mathcal{B}} = ((\mathcal{B}_i)_1^n; \mathcal{Q}; \mathcal{B}_0) : \tilde{r} = ((r_i)_0^n; r).$$

Then $\tilde{\mathcal{I}} = ((\mathcal{I}_i)_1^n; \mathcal{K}; \mathcal{I}_0) = \mathrm{Im}(U\tilde{f} - U\tilde{g})$ is an ideal in $\tilde{\mathcal{B}}$. In fact it suffices to take in the source of action map (1.17) one of the $\otimes$-arguments equal to $(\tilde{f} - \tilde{g})x$ for $x \in \mathcal{A}_i$ or $\mathcal{P}$ or $\mathcal{A}_0$. Then

$$\begin{aligned}
&\alpha(\cdots \otimes b_i^p \otimes \cdots \otimes (\tilde{f} - \tilde{g})x \otimes \cdots \otimes q_j \otimes \cdots \otimes b_0) \\
&= \alpha(\cdots \otimes f_i r_i b_i^p \otimes \cdots \otimes \tilde{f}x \otimes \cdots \otimes frq_j \otimes \cdots \otimes f_0 r_0 b_0) \\
&- \alpha(\cdots \otimes g_i r_i b_i^p \otimes \cdots \otimes \tilde{g}x \otimes \cdots \otimes grq_j \otimes \cdots \otimes g_0 r_0 b_0) \\
&= (f-g)\alpha(\cdots \otimes r_i b_i^p \otimes \cdots \otimes x \otimes \cdots \otimes rq_j \otimes \cdots \otimes r_0 b_0) \in \mathcal{K}.
\end{aligned}$$

Thus $\tilde{\mathcal{B}}/\tilde{\mathcal{I}}$ is an $n \wedge 1$-operad module and the rest of the proof goes similarly to the case of operads. □

1.27 Corollary. *The category ${}_n\mathrm{Op}_1$ is complete and cocomplete.*

Proof. In fact, $\mathcal{D} = \mathbf{dg}^S$ is complete and cocomplete. Completeness of ${}_n\mathrm{Op}_1 \simeq \mathcal{D}^\top$ follows by [BW05, Corollary 3.4.3]. We have seen that the category ${}_n\mathrm{Op}_1$ has coequalizers. Therefore ${}_n\mathrm{Op}_1$ is cocomplete by [BW05, Corollary 9.3.3]. □

1.28. Peculiar features. The category $(\mathcal{A}_1,\dots,\mathcal{A}_n)$-mod-$\mathcal{B}$ of operad $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})$-modules is a subcategory of ${}_n\mathrm{Op}_1$, whose objects are $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ (shortly $\mathcal{P}$) and morphisms are $(1_{\mathcal{A}_1},\dots,1_{\mathcal{A}_n};h;1_{\mathcal{B}})$. It has an initial object $\mathcal{I}=\mathcal{B}(0)$ with

$$\mathcal{I}(k)=\begin{cases}\mathcal{B}(0), & \text{if } k=0\\ 0, & \text{if } k\neq 0\end{cases}, \qquad k\in\mathbb{N}^n.$$

Actions are given by

$$\lambda_0^i=\mathrm{id}:\mathcal{B}(0)\to\mathcal{B}(0),$$
$$\rho=\mu^{\mathcal{B}}_{0,\dots,0}:(\otimes^{\mathbf{m}}\mathcal{B}(0))\otimes\mathcal{B}(m)\to\mathcal{B}(0).$$

A *system of relations* in an operad $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})$-module $\mathcal{P}$ means a set (of relations) R, arity function $a:R\to\mathbb{N}^n$, grade function $g:R\to\mathbb{Z}$ and for each $r\in R$ elements $x_r,y_r\in\mathcal{P}(a(r))^{g(r)}$ supposed to be identified by the relation $x_r=y_r$. A system of relations gives rise to a free graded $\Bbbk$-module $\Bbbk R\in\mathbf{gr}^{\mathbb{N}^n}$ with

$$\Bbbk R(a)^g=\Bbbk\{r\in R\mid a(r)=a,\ g(r)=g\},\quad \text{for } a\in\mathbb{N}^n,\ g\in\mathbb{Z},$$

and to a map $\Bbbk R\to\mathcal{P}$, $r\mapsto x_r-y_r$. Denote by $\mathcal{N}$ the image of this map in abelian category $\mathbf{gr}^{\mathbb{N}^n}$. Let $\mathcal{K}=(0,\dots,0;\mathcal{K};0)$ be the graded ideal of $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ generated by $\mathcal{N}$. If $\mathcal{N}\partial\subset\mathcal{K}$, then $\mathcal{K}$ is a differential graded ideal and the quotient $\mathcal{P}/\mathcal{K}$ in $\mathbf{dg}^{\mathbb{N}^n}$ is an operad $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{B})$-module. This is the quotient of an operad module by a system of relations.

1.28.1. Induced modules. Assume that $f_i:\mathcal{A}_i\to\mathcal{B}_i$ are morphisms of operads for $i\in[n]$. For any $n\wedge 1$-operad module $((\mathcal{A}_i)_{i\in[n]};\mathcal{P})$ there is a $(\mathcal{B}_i)_{i\in[n]}$-module $\mathcal{Q}=\bigodot_{i=0}^n\mathcal{B}_i\odot^i_{\mathcal{A}_i}\mathcal{P}$, which is the colimit of the diagram (the coequalizer)

$$\begin{tikzcd}
\odot_{\geqslant0}((\mathcal{B}_i)_{i\in[n]};\odot_{\geqslant0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P})) \arrow[r, "\odot_{\geqslant0}({}^{[n]}1;\alpha)"] \arrow[d, "\odot_{\geqslant0}({}^{[n]}1;\odot_{\geqslant0}((f_i)_{i\in[n]};1))"'] & \odot_{\geqslant0}((\mathcal{B}_i)_{i\in[n]};\mathcal{P}) \\
\odot_{\geqslant0}((\mathcal{B}_i)_{i\in[n]};\odot_{\geqslant0}((\mathcal{B}_i)_{i\in[n]};\mathcal{P})) \arrow[r, "\sim"] \arrow[ur, "\mu"] \arrow[ur, phantom, shift right=3, "="] & \odot_{\geqslant0}((\mathcal{B}_i\odot\mathcal{B}_i)_{i\in[n]};\mathcal{P}) \arrow[u, "\odot_{\geqslant0}({}^{[n]}\mu;1)"']
\end{tikzcd}$$

This can be described also using monads $\mathbb{A}:\mathcal{X}\mapsto\odot_{\geqslant0}((\mathcal{A}_i)_{i\in[n]};\mathcal{X})$, $\mathbb{B}:\mathcal{X}\mapsto\odot_{\geqslant0}((\mathcal{B}_i)_{i\in[n]};\mathcal{X})$ in $\mathbf{dg}^{\mathbb{N}^n}$ and a morphism $f=\odot_{\geqslant0}((f_i)_{i\in[n]};1):\mathbb{A}\to\mathbb{B}$ between them. Denote $\alpha^{\mathcal{P}}:\mathbb{A}\mathcal{P}\to\mathcal{P}$ the action on the $\mathbb{A}$-module $\mathcal{P}$. The induced module $\mathcal{Q}$ is the colimit $\mathbb{B}_{\mathbb{A}}\mathcal{P}$ of the diagram in the category of $\mathbb{B}$-modules (the coequalizer of)

$$\begin{tikzcd}
\mathbb{B}\mathbb{A}\mathcal{P} \arrow[rr, "\mathbb{B}\alpha^{\mathcal{P}}"] \arrow[dr, "\mathbb{B}f"'] & & \mathbb{B}\mathcal{P} \\
& \mathbb{B}\mathbb{B}\mathcal{P} \arrow[ur, "\mu^{\mathbb{B}}"'] &
\end{tikzcd}$$

If $\mathcal{P} = \odot_{\geqslant 0}(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{X}; \mathcal{A}_0)$ is a free $n \wedge 1$-module, then the $n \wedge 1$-module $\bigodot_{i=1}^n \mathcal{B}_i \odot_{\mathcal{A}_i}^i \mathcal{P} \odot_{\mathcal{A}_0}^0 \mathcal{B}_0 = \odot_{\geqslant 0}(\mathcal{B}_1, \dots, \mathcal{B}_n; \mathcal{X}; \mathcal{B}_0)$ is free as well. It is also generated by $\mathcal{X}$. In fact, the coequalizer in question has the structure of a contractible coequalizer [BW05, Definition 3.3.3] with $d^1 = \mathbb{B}f \cdot \mu^{\mathbb{B}} : \mathbb{BAA}\mathcal{X} \to \mathbb{BA}\mathcal{X}$ and $d = \mathbb{B}f \cdot \mu^{\mathbb{B}} : \mathbb{BA}\mathcal{X} \to \mathbb{B}\mathcal{X}$

$$\begin{tikzcd}
\mathbb{BAA}\mathcal{X} \arrow[rr, shift left, "d^0=\mathbb{B}\mu^{\mathbb{A}}"] \arrow[dr, "\mathbb{B}f"'] & & \mathbb{BA}\mathcal{X} \arrow[ll, shift left, "t=\mathbb{BA}\eta"] \arrow[rr, "\mathbb{B}f"] & & \mathbb{BB}\mathcal{X} \arrow[d, "\mu^{\mathbb{B}}"] \\
& \mathbb{BBA}\mathcal{X} \arrow[ur, "\mu^{\mathbb{B}}"'] & & & \mathbb{B}\mathcal{X} \arrow[ull, "s=\mathbb{B}\eta"]
\end{tikzcd}$$

The following is a part of the theory of modules in general categories, not necessarily linear.

1.29 Proposition. *So defined functor* $(\mathcal{A}_i)_{i\in[n]}$-mod $\to (\mathcal{B}_i)_{i\in[n]}$-mod, $\mathcal{P} \mapsto \bigodot_{i=0}^n \mathcal{B}_i \odot_{\mathcal{A}_i}^i \mathcal{P}$, *is left adjoint to the base change functor* $(\mathcal{B}_i)_{i\in[n]}$-mod $\to (\mathcal{A}_i)_{i\in[n]}$-mod, $\mathcal{R} \mapsto {}_{f_1,\dots,f_n}\mathcal{R}_{f_0}$. *Thus, the mapping*

$$(\mathcal{B}_i)_{i\in[n]}\text{-mod}(\bigodot_{i=0}^n \mathcal{B}_i \odot_{\mathcal{A}_i}^i \mathcal{P}, \mathcal{R}) \xrightarrow{\sim} (\mathcal{A}_i)_{i\in[n]}\text{-mod}(\mathcal{P}, {}_{f_1,\dots,f_n}\mathcal{R}_{f_0}), \qquad h \mapsto \eta \cdot h,$$

is a bijection, where $\eta : \mathcal{P} \to \bigodot_{i=0}^n \mathcal{B}_i \odot_{\mathcal{A}_i}^i \mathcal{P}$ *comes from the unit of the monad* $\odot_{\geqslant 0}((\mathcal{B}_i); -)$.

Proof. Since $\mathcal{Q} = \bigodot_{i=0}^n \mathcal{B}_i \odot_{\mathcal{A}_i}^i \mathcal{P}$ is a colimit (a coequalizer), the following set of morphisms is a limit (an equalizer of a pair of maps):

$$\begin{aligned}
&(\mathcal{B}_i)_{i\in[n]}\text{-mod}(\mathcal{Q}, \mathcal{R}) \\
&= \lim \left[\begin{tikzcd}
(\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mathcal{B}_i); \mathcal{P}), \mathcal{R}) \arrow[r, "(\odot_{\geqslant 0}((1);\alpha),1)"] \arrow[d, "(\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mu_i);1),1)"'] & (\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mathcal{B}_i); \odot_{\geqslant 0}((\mathcal{A}_i); \mathcal{P})), \mathcal{R}) \\
(\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mathcal{B}_i \odot \mathcal{B}_i); \mathcal{P}), \mathcal{R}) \arrow[r, "\sim"] & (\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((\mathcal{B}_i); \odot_{\geqslant 0}((\mathcal{B}_i); \mathcal{P})), \mathcal{R}) \arrow[u, "(\mathcal{B}_i)\text{-mod}(\odot_{\geqslant 0}((1);\odot_{\geqslant 0}((f_i);1)),1)"']
\end{tikzcd} \right] \\
&= \lim \left[\begin{tikzcd}
\mathbf{dg}^{\mathbb{N}^n}(\mathcal{P}, \mathcal{R}) \arrow[r, "\mathbf{dg}^{\mathbb{N}^n}(\alpha,1)"] \arrow[d, "\odot_{\geqslant 0}((\mathcal{B}_i);-)"'] & \mathbf{dg}^{\mathbb{N}^n}(\odot_{\geqslant 0}((\mathcal{A}_i); \mathcal{P}), \mathcal{R}) \\
\mathbf{dg}^{\mathbb{N}^n}(\odot_{\geqslant 0}((\mathcal{B}_i); \mathcal{P}), \odot_{\geqslant 0}((\mathcal{B}_i); \mathcal{R})) \arrow[r, "\mathbf{dg}^{\mathbb{N}^n}(1,\alpha)"] & \mathbf{dg}^{\mathbb{N}^n}(\odot_{\geqslant 0}((\mathcal{B}_i); \mathcal{P}), \mathcal{R}) \arrow[u, "\mathbf{dg}^{\mathbb{N}^n}(\odot_{\geqslant 0}((f_i);1),1)"']
\end{tikzcd} \right].
\end{aligned}$$

Thus, a morphism $u : \mathcal{P} \to \mathcal{R} \in \mathbf{dg}^{\mathbb{N}^n}$ belongs to the equalizer iff

$$\begin{aligned}
\big[\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]}; \mathcal{P}) \xrightarrow{\odot_{\geqslant 0}((f_i);1)} \odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \mathcal{P}) \xrightarrow{\odot_{\geqslant 0}((1);u)} \odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \mathcal{R}) \xrightarrow{\alpha} \mathcal{R}\big] \\
= \big[\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]}; \mathcal{P}) \xrightarrow{\alpha} \mathcal{P} \xrightarrow{u} \mathcal{R}\big].
\end{aligned}$$

Equivalently, the mapping $u : \mathcal{P} \to {}_{f_1,\dots,f_n}\mathcal{R}_{f_0}$ is a homomorphism of $(\mathcal{A}_i)_{i\in[n]}$-modules. $\square$

Together with Lemma 1.34 this proposition implies the following

1.30 Corollary. *In assumptions of Section 1.28.1 denote* $\mathcal{Q} = \bigodot_{i=0}^{n} \mathcal{B}_i \odot_{\mathcal{A}_i}^{i} \mathcal{P}$*. Then the diagram in the category* ${}_n\mathrm{Op}_1$

$$\begin{tikzcd}
(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{A}_0(0); \mathcal{A}_0) \arrow[r, "(1,\dots,1;\rho_\varnothing;1)"] \arrow[d, "(f_1,\dots,f_n;f_0(0);f_0)"'] & (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{A}_0) \arrow[d, "(f_1,\dots,f_n;\eta;f_0)"] \\
(\mathcal{B}_1, \dots, \mathcal{B}_n; \mathcal{B}_0(0); \mathcal{B}_0) \arrow[r, "(1,\dots,1;\rho_\varnothing;1)"] & (\mathcal{B}_1, \dots, \mathcal{B}_n; \mathcal{Q}; \mathcal{B}_0)
\end{tikzcd}$$

is a pushout square.

1.31. The operad $T\mathcal{N}\sqcup\mathcal{O}$**.** As explained in [Lyu11, Proposition 1.8] the operad $T\mathcal{N}\sqcup\mathcal{O}$ is a direct sum over ordered rooted trees t with inputs whose vertices are coloured with $\mathcal{N}$ and $\mathcal{O}$ that are terminal the following sense. Two conditions hold:

$\lightning$) t contains no edge whose both ends are coloured with $\mathcal{O}$;

$\lightning\lightning$) insertion of a unary vertex coloured with $\mathcal{O}$ into an arbitrary edge of t breaks condition $\lightning$).

By the way, these conditions imply that t contains no edge whose both ends are coloured with $\mathcal{N}$. The second condition is related to inserting a unit $\eta : \Bbbk \to T\mathcal{N} \sqcup \mathcal{O}$ identified with the unit $\eta : \Bbbk \to \mathcal{O}$ in an arbitrary summand of $T\mathcal{N} \sqcup \mathcal{O}$ represented by a coloured tree. The first condition means that one can not apply binary composition in $\mathcal{O}$ inside an element of

$$C(\mathcal{N}, \mathcal{O}; t) \simeq \otimes^{v\in(\mathrm{v}(t),\leqslant)} c(v)(|v|) \subset C(\mathcal{N} \oplus \mathcal{O}; t) \subset T(\mathcal{N} \oplus \mathcal{O})$$

represented by t. Here $\leqslant$ is an admissible order on $\mathrm{v}(t)$, the set of internal vertices of an ordered rooted trees t with inputs $\mathrm{Inp}\, t$, see Section 1.5 of [Lyu11]. The set $\mathrm{v}(t)$ is coloured by the function $c : \mathrm{v}(t) \to \{\mathcal{N}, \mathcal{O}\}$, which singles out an individual summand of $(\mathcal{N} \oplus \mathcal{O})^{\otimes \mathrm{v}(t)}$.

Any sequence of contractions of edges whose ends are coloured with $\mathcal{O}$ and insertions of unary vertices coloured with $\mathcal{O}$ applied to given tree t' may lead to no more than one terminal tree t. The mapping $t' \mapsto t$ is well defined. The summand $C(\mathcal{N}, \mathcal{O}; t')$ of $T(\mathcal{N}\oplus\mathcal{O})$ corresponding to t' is mapped by binary compositions in $\mathcal{B}$ and insertions of the unit of $\mathcal{B}$ to the summand $C(\mathcal{N}, \mathcal{O}; t)$. Associativity and unitality of $\mathcal{O}$ implies that this map is unique.

The requirement of $\mathcal{O} \to C = T\mathcal{N}\sqcup\mathcal{O}$ being a morphism of operads (see Corollary A.4) reduces to agreeing with binary compositions and units. Therefore, in the case of operads equation (A.10) is equivalent to a family of similar diagrams with $\alpha : T\mathcal{O} \to \mathcal{O}$ replaced with the unit $1_\mathcal{O} : \Bbbk \to \mathcal{O}$ and with the binary compositions $\mathcal{O} \odot \mathcal{O} \supset \Bbbk \otimes \cdots \otimes \Bbbk \otimes \mathcal{O} \otimes \Bbbk \otimes \cdots \otimes \Bbbk \otimes \mathcal{O} \xrightarrow{\mu} \mathcal{O}$. That is, the kernel $\mathcal{J}$ of $T(\mathcal{N} \oplus \mathcal{O}) \to T\mathcal{N} \sqcup \mathcal{O}$ is generated as an ideal by relations coming from binary composition in $\mathcal{O}$ (corresponding to contraction of an $\mathcal{O}$-coloured edge) and from inserting a unit of $\mathcal{O}$ (corresponding to insertion of $\mathcal{O}$-coloured unary vertex into an edge). Together with the above this implies that an element of

$C(\mathcal{N},\mathcal{O};t')$ is equivalent to a unique element of $C(\mathcal{N},\mathcal{O};t)$ modulo $\mathcal{I}$. In fact, for elements $x'\in C(\mathcal{N},\mathcal{O};t')$ and $x''\in C(\mathcal{N},\mathcal{O};t'')$ related by an elementary relation as above this holds true since there is either a path $(t',t'',\dots,t)$ or a path $(t'',t',\dots,t)$ consisting of contractions or unary insertions. We conclude that [Lyu11, eq. (1.2)]

$$T\mathcal{N}\sqcup\mathcal{O}=\bigoplus_{t\in\mathcal{L}}C(\mathcal{N},\mathcal{O};t),$$

where $\mathcal{L}$ is the list of terminal trees.

1.32 Proposition. *The category ${}_n\mathrm{Op}_1$ of $n\wedge 1$-operad modules with quasi-isomorphisms as weak equivalences and degreewise surjections as fibrations is a model category.*

Proof. Let us apply Theorem 0.9 to the adjunction $F:\mathbf{dg}^S\leftrightarrows {}_n\mathrm{Op}_1:U$, where $S=n\mathbb{N}\sqcup\mathbb{N}^n\sqcup\mathbb{N}$. By Corollary 1.27 the category ${}_n\mathrm{Op}_1$ is complete and cocomplete.

Let $\mathcal{N}\in\operatorname{Ob}\mathbf{dg}^{\mathbb{N}}$ or $\mathcal{N}\in\operatorname{Ob}\mathbf{dg}^{\mathbb{N}^n}$ be a complex. For instance, $\mathcal{N}=\mathbb{K}_x[-p]$ for $p\in\mathbb{Z}$, $x\in\mathbb{N}$ or $x\in\mathbb{N}^n$. Let $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ be an $n\wedge 1$-operad module. Denote by $\tilde{\mathcal{N}}$ the object $(0,\dots,0;0;\mathcal{N})$ or $(0,\dots,0;\mathcal{N};0)$ or $(0,\dots,0,\mathcal{N},0,\dots,0;0;0)$ ($\mathcal{N}$ on i^{th} place) of $\mathbf{dg}^S$. We shall prove case by case that if $\mathcal{N}$ is contractible then so is $U(F\tilde{\mathcal{N}}\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B}))$.

By Corollary A.4 the operad module in question $F\tilde{\mathcal{N}}\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ is the quotient of $F(\tilde{\mathcal{N}}\oplus(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B}))$ by the smallest ideal $\mathcal{I}$ generated by relations which tell that $(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})\to F(\tilde{\mathcal{N}}\oplus(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B}))/\mathcal{I}$ is a morphism of $n\wedge 1$-operad modules. Notice by the way that

$$\begin{aligned}
F(0,\dots,0;0;\mathcal{N})\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})&=(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{Q};T\mathcal{N}\sqcup\mathcal{B}),\\
\mathcal{Q}&=\mathcal{P}\odot^0_{\mathcal{B}}(T\mathcal{N}\sqcup\mathcal{B}),\\
F(0,\dots,0;\mathcal{N};0)\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})&=(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{R};\mathcal{B}),\\
\mathcal{R}&=\mathcal{P}\oplus\odot_{\geqslant 0}(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{N};\mathcal{B}),\\
F(0,\dots,0,\mathcal{N},0,\dots,0;0;0)\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})&=(\mathcal{A}_1,\dots,T\mathcal{N}\sqcup\mathcal{A}_i,\dots,\mathcal{A}_n;\mathcal{S};\mathcal{B}),\\
\mathcal{S}&=(T\mathcal{N}\sqcup\mathcal{A}_i)\odot^i_{\mathcal{A}_i}\mathcal{P}.
\end{aligned}$$

More important is the presentation of operad polymodules as direct sums over some kind of trees. This presentation we use for the proof. Similarly to the operad case considered in Section 1.31 we find that

$$\mathcal{Q}=\oplus_{\tau\in\mathcal{L}_{\mathcal{Q}}}\mathcal{Q}(\mathcal{P},\mathcal{N},\mathcal{B};\tau),\qquad\text{where}\qquad \mathcal{Q}(\mathcal{P},\mathcal{N},\mathcal{B};\tau)=\operatorname*{colim}_{\leqslant\in\mathcal{G}_{\mathcal{Q}}(\tau)}\mathcal{Q}(\mathcal{P},\mathcal{N},\mathcal{B};\tau)(\leqslant).$$

Here objects of the groupoid $\mathcal{G}_{\mathcal{Q}}(\tau)$ are admissible (compatible with the natural partial order $\preccurlyeq$ with the root as the biggest element) total orderings $\leqslant$ of the set of all vertices $\overline{\mathrm{v}}(\tau)$. By definition between any two objects of $\mathcal{G}(\tau)$ there is precisely one morphism. Thus $\mathcal{G}(\tau)$ is contractible (equivalent to the terminal category with one object and one morphism). Therefore the colimit is isomorphic to any of

$$\mathcal{Q}(\mathcal{P},\mathcal{N},\mathcal{B};\tau)(\leqslant)=\otimes^{v\in(\overline{\mathrm{v}}(\tau),\leqslant)}c(v)(|v|).$$

Here $\tau = (\tau, c, |\cdot|)$ is an ordered rooted tree τ with inputs $\operatorname{Inp}\tau$, a colouring $c : \overline{\mathrm{v}}(\tau) \to \{\mathcal{P}, \mathcal{N}, \mathcal{B}\}$ such that $c(\operatorname{Inp}\tau) \subset \{\mathcal{P}\}$, $c(\mathrm{v}(\tau)) \subset \{\mathcal{N}, \mathcal{B}\}$ and an arbitrary function $|\cdot| : \operatorname{Inp}(\tau) \to \mathbb{N}^n$, which complements the valency $|\cdot| : \mathrm{v}(\tau) \to \mathbb{N}$. The set $\mathcal{L}_{\mathcal{Q}}$ of terminal trees consists of τ such that

$*$) τ contains no edge whose both ends are coloured with $\mathcal{B}$;

$**$) τ contains no vertex coloured with $\mathcal{B}$ whose all entering edges have other ends coloured with $\mathcal{P}$;

$***$) insertion of a unary vertex coloured with $\mathcal{B}$ into an arbitrary edge of τ breaks condition $*$) or $**$).

Respectively,

$$\mathcal{S} = \oplus_{\tau \in \mathcal{L}_{\mathcal{S}}} \mathcal{S}(\mathcal{N}, \mathcal{A}_i, \mathcal{P}; \tau), \qquad \text{where} \qquad \mathcal{S}(\mathcal{N}, \mathcal{A}_i, \mathcal{P}; \tau) = \operatorname*{colim}_{\leqslant \in \mathcal{G}_{\mathcal{S}}(\tau)} \mathcal{S}(\mathcal{N}, \mathcal{A}_i, \mathcal{P}; \tau)(\leqslant).$$

The colimit over the contractible groupoid $\mathcal{G}_{\mathcal{S}}(\tau)$ is isomorphic to expression under colimit in any vertex $\leqslant$. Assuming that $\ell \in \mathbb{N}^n$, $\ell^i = |\operatorname{Inp}\tau|$, we have

$$\mathcal{S}(\mathcal{N}, \mathcal{A}_i, \mathcal{P}; \tau)(\leqslant)(\ell) = \left[\otimes^{v \in (\mathrm{v}(\tau) - \{\mathrm{root}\}, \leqslant)} c(v)(|v|)\right] \otimes \mathcal{P}(\ell, \ell^i \mapsto q).$$

Here $\tau = (\tau, c)$ is an ordered rooted tree τ with inputs $\operatorname{Inp}\tau$ and a colouring $c : \mathrm{v}(\tau) - \{\mathrm{root}\} \to \{\mathcal{N}, \mathcal{A}_i\}$. The set $\mathcal{L}_{\mathcal{S}}$ of terminal trees consists of τ such that

$*$) τ contains no edge whose both ends are coloured with $\mathcal{B}$;

$**$) τ contains no edge adjacent to the root whose one end is coloured with $\mathcal{B}$;

$***$) insertion of a unary vertex coloured with $\mathcal{B}$ into an arbitrary edge of τ breaks condition $*$) or $**$).

Let us prove existence of contracting homotopy similarly to the case of operads [Lyu11, Proposition 1.8]. Let $\mathcal{N} \in \operatorname{Ob}\mathbf{dg}^{\mathbb{N}}$ or $\mathcal{N} \in \operatorname{Ob}\mathbf{dg}^{\mathbb{N}^n}$ be contractible and let $h : \mathcal{N} \to \mathcal{N}$ be a contracting homotopy, $\deg h = -1$, $dh + hd = 1_{\mathcal{N}}$. Let us show that the operad module morphism $\alpha = \mathrm{in}_2 : \mathcal{M} = (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to F\tilde{\mathcal{N}} \sqcup (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$ is homotopy invertible. Consider the operad module morphism $\beta : F\tilde{\mathcal{N}} \sqcup (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$ which restricts to $\beta\big|_{(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})} = \mathrm{id}$ and $\beta\big|_{F\tilde{\mathcal{N}}}$, adjunct to $0 : \tilde{\mathcal{N}} \to U(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$. Then $\alpha \cdot \beta = \mathrm{id}$ and $g = \beta \cdot \alpha$ is homotopic to $f = \mathrm{id}_{F\tilde{\mathcal{N}} \sqcup (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})}$ in the $\mathbf{dg}$-category $\mathbf{dg}^S$, as we show next. The homotopy h is extended by 0 to the map $h' = h \oplus 0 : \tilde{\mathcal{N}} \oplus U(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to \tilde{\mathcal{N}} \oplus U(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$, which satisfies $dh' + h'd = f| - g| : \tilde{\mathcal{N}} \oplus U(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to \tilde{\mathcal{N}} \oplus U(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$. In all three cases the endomorphisms f, g of $F\tilde{\mathcal{N}} \sqcup (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$ lift to endomorphisms of $F(\tilde{\mathcal{N}} \oplus (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}))$ obtained by applying $f\big|_{\tilde{\mathcal{N}}} = 1 : \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}$, $f\big|_{\mathcal{M}} = 1 : \mathcal{M} \to \mathcal{M}$, $g\big|_{\tilde{\mathcal{N}}} = 0 : \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}$, $g\big|_{\mathcal{M}} = 1 : \mathcal{M} \to \mathcal{M}$ to each $\otimes$-factor corresponding to a vertex of

the tree. For an arbitrary pair of trees (t,τ) choose admissible orderings $(\leqslant,\leqslant)$. Then the summands of $F\tilde{\mathcal{N}}\sqcup(\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})$ are preserved by f and g and the restriction to the summand is $f\otimes\cdots\otimes f$ and $g\otimes\cdots\otimes g$ respectively. Define a $\Bbbk$-endomorphism $\hat{h}=\sum_{v\in(\mathrm{v}(t),\leqslant)}f\otimes\cdots\otimes f\otimes h'\otimes g\otimes\cdots\otimes g$ of degree -1, where h' is applied on place indexed by v. Then

$$d\hat{h}+\hat{h}d=\sum_{v\in(\mathrm{v}(t),\leqslant)}f\otimes\cdots\otimes f\otimes(f-g)\otimes g\otimes\cdots\otimes g=f\otimes\cdots\otimes f-g\otimes\cdots\otimes g=f-g.$$

Therefore, f and g are homotopic to each other and α is homotopy invertible. □

Proposition 1.29 gives a recipe of computing colimits in the category ${}_n\mathrm{Op}_1$ of $n\wedge 1$-operad modules in two steps. First of all compute colimits $\mathcal{B}_i$ on each of $n+1$ operadic places $i\in[n]$. Take induced module over $(\mathcal{B}_1,\dots,\mathcal{B}_n;\mathcal{B}_0)$ on each node of the diagram and consider the obtained diagram in the category $(\mathcal{B}_1,\dots,\mathcal{B}_n)$-mod-$\mathcal{B}_0$. Secondly find the colimit of the latter diagram, by finding its colimit in $\mathbf{dg}^{\mathbb{N}^n}$, then generating by it a free $(\mathcal{B}_1,\dots,\mathcal{B}_n;\mathcal{B}_0)$-module F, dividing it precisely by such relations that canonical mapping from any module to the quotient of F were a morphism of $(\mathcal{B}_1,\dots,\mathcal{B}_n;\mathcal{B}_0)$-modules.

1.33. Some colimits of operad modules.

1.34 Lemma. *Let $f_i:\mathcal{A}_i\to\mathcal{B}_i$ be morphisms of operads for $i\in[n]$. Let $\mathcal{P}$ be a $(\mathcal{B}_i)_{i\in[n]}$-module. Then there is a unique morphism ϕ such that*

$$((f_i)_{i\in[n]};\phi):((\mathcal{A}_i)_{i\in[n]};\mathcal{A}_0(0))\to((\mathcal{B}_i)_{i\in[n]};\mathcal{P})\in{}_n\mathrm{Op}_1\,.$$

Proof. The morphism ϕ is recovered from the equation

$$\big(\mathcal{A}_0(0)\xrightarrow[1]{\rho_\varnothing^{\mathcal{A}_0(0)}}\mathcal{A}_0(0)\xrightarrow{\phi}\mathcal{P}(0)\big)=\big(\mathcal{A}_0(0)\xrightarrow{f_0(0)}\mathcal{B}_0(0)\xrightarrow{\rho_\varnothing^{\mathcal{P}}}\mathcal{P}(0)\big)$$

in the unique possible form $\phi=f_0(0)\cdot\rho_\varnothing^{\mathcal{P}}$. It is compatible with the action ρ because the diagram

$$\begin{tikzcd}
\mathcal{A}_0(0)^{\otimes m}\otimes\mathcal{A}_0(m) \arrow[r, "f_0(0)^{\otimes m}\otimes f_0(m)"] \arrow[d, "\mu^{\mathcal{A}_0}"'] \arrow[dr, phantom, "="] & \mathcal{B}_0(0)^{\otimes m}\otimes\mathcal{B}_0(m) \arrow[r, "\rho_\varnothing^{\otimes m}\otimes 1"] \arrow[d, "\mu^{\mathcal{B}_0}"'] \arrow[dr, phantom, "="] & \mathcal{P}_0(0)^{\otimes m}\otimes\mathcal{B}_0(m) \arrow[d, "\rho^m_{m0}"] \\
\mathcal{A}_0(0) \arrow[r, "f_0(0)"] & \mathcal{B}_0(0) \arrow[r, "\rho_\varnothing"] & \mathcal{P}_0(0)
\end{tikzcd}$$

commutes. For the second square this follows by associativity of the action. □

1.35 Corollary. *For arbitrary operads $\mathcal{C}_i$, $\mathcal{A}_i$, $i\in[n]$, there is an isomorphism in ${}_n\mathrm{Op}_1$*

$$((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))\sqcup((\mathcal{A}_i)_{i\in[n]};\mathcal{A}_0(0))\simeq((\mathcal{C}_i\sqcup\mathcal{A}_i)_{i\in[n]};(\mathcal{C}_0\sqcup\mathcal{A}_0)(0)).$$

1.36 Proposition. *Let $A = (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{A}_0)$ be an $n \wedge 1$-operad module. Let $\beta_i : \mathcal{M}_i \to \mathcal{A}_i \in \mathbf{dg}^{\mathbb{N}}$ be chain maps for $i \in [n]$. Denote $M = (\mathcal{M}_1, \dots, \mathcal{M}_n; 0; \mathcal{M}_0)$ and $\beta = (\beta_1, \dots, \beta_n; 0; \beta_0) : (\mathcal{M}_1, \dots, \mathcal{M}_n; 0; \mathcal{M}_0) \to (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{A}_0)$. Then $A\langle M, \beta\rangle$ defined in diagram (0.3) is isomorphic to $((\mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle)_{i\in[n]}; \bigodot_{i=0}^n \mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle \odot^i_{\mathcal{A}_i} \mathcal{P})$.*

Proof. Denote $\mathcal{R} = \bigodot_{i=0}^n \mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle \odot^i_{\mathcal{A}_i} \mathcal{P}$ and $\mathcal{C}_i = T(\mathcal{M}_i[1])$. As we know from [Lyu11, Section 1.10] in $\mathscr{G} = {}_n\mathrm{Op}_1^{\mathbf{gr}}$ the graded module underlying $A\langle M, \beta\rangle$ is isomorphic to $((\mathcal{C}_i)_{i\in[n]}; \mathcal{C}_0(0)) \sqcup ((\mathcal{A}_i)_{i\in[n]}; \mathcal{P})$. Clearly, this coproduct is also a colimit of the following diagram (a pushout)

$$\begin{tikzcd}
& ((\mathcal{A}_i)_{i\in[n]}; \mathcal{A}_0(0)) \arrow[r, "((1);!)"] \arrow[d, "\mathrm{in}_2"'] & ((\mathcal{A}_i)_{i\in[n]}; \mathcal{P}) \arrow[d, "\mathrm{in}_2"] \\
((\mathcal{C}_i \sqcup \mathcal{A}_i)_{i\in[n]}; (\mathcal{C}_0 \sqcup \mathcal{A}_0)(0)) = & ((\mathcal{C}_i)_{i\in[n]}; \mathcal{C}_0(0)) \sqcup ((\mathcal{A}_i)_{i\in[n]}; \mathcal{A}_0(0)) \arrow[r, "1\sqcup((1);!)"] & A\langle M, \beta\rangle
\end{tikzcd}$$

Here the equation is due to Corollary 1.35. Denote $\mathcal{B}_i = \mathcal{C}_i \sqcup \mathcal{A}_i \simeq \mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle$ in $\mathrm{Op}^{\mathbf{gr}}$. Applying Corollary 1.30 to the canonical embeddings $\mathrm{in}_2 : \mathcal{A}_i \to \mathcal{B}_i$ we deduce an isomorphism $\psi_3 : A\langle M, \beta\rangle \to ((\mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle)_{i\in[n]}; \bigodot_{i=0}^n \mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle \odot^i_{\mathcal{A}_i} \mathcal{P})$ in ${}_n\mathrm{Op}_1^{\mathbf{gr}}$.

In order to show that the isomorphism is actually in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$ we consider diagram on the next page, where $\phi_k = \psi_k^{-1}$, $1 \leqslant k \leqslant 3$. Notice that the isomorphism ψ_2 is nothing else but the isomorphism ψ_3, written for the operad module $((\mathcal{A}_i)_{i\in[n]}; \mathcal{P})$ instead of $\mathcal{P}$, taking into account that

$$\bigodot_{i=0}^n \mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle \odot^i_{\mathcal{A}_i} \odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]}; \mathcal{P})) \simeq \odot_{\geqslant 0}((\mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle)_{i\in[n]}; \mathcal{P}),$$

see Section 1.28.1. The isomorphism ψ_1 follows from the fact that $\bar{F} : {}_n\mathrm{Op}_1^{\mathbf{gr}} \to \mathbf{gr}^{n\mathbb{N}\sqcup\mathbb{N}^n\sqcup\mathbb{N}}$ preserves colimits. The map $z : \odot_{\geqslant 0}((\mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle)_{i\in[n]}; \mathcal{P}) \to \bigodot_{i=0}^n \mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle \odot^i_{\mathcal{A}_i} \mathcal{P}$ is the canonical projection.

We claim that the diagram commutes. Its two top squares are in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$, while the bottom vertical isomorphisms are constructed only in $\mathscr{G} = {}_n\mathrm{Op}_1^{\mathbf{gr}}$. Thus, squares $\boxed{3}$ and $\boxed{4}$ make sense in $\mathscr{G}$. First of all, square $\boxed{1}$ commutes due to definition (0.3) of $\mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle$. Commutativity of square $\boxed{2}$ follows from the equation

$$\begin{tikzcd}
\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]}; \mathcal{P}) \arrow[r, "\alpha"] \arrow[d, "\odot_{\geqslant 0}((\bar{\jmath}_i)_{i\in[n]};1)"'] \arrow[dr, phantom, "="] & \mathcal{P} \arrow[d, "\eta"] \\
\odot_{\geqslant 0}((\mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle)_{i\in[n]}; \mathcal{P}) \arrow[r, "z"] & \bigodot_{i=0}^n \mathcal{A}_i\langle\mathcal{M}_i, \beta_i\rangle \odot^i_{\mathcal{A}_i} \mathcal{P}
\end{tikzcd} \tag{1.19}$$

proven directly from the definition of the induced operad module. Namely paths in the following diagram

$$\begin{tikzcd}
\odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]}; \mathcal{P})) \arrow[d, "\odot_{\geqslant 0}((1);\odot_{\geqslant 0}((\bar{\jmath}_i);1))"'] \arrow[drr, "\odot_{\geqslant 0}((1);\alpha)"'] & \odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]}; \mathcal{P}) \arrow[l, "\odot_{\geqslant 0}((\eta);1)"'] \arrow[r, "\alpha"] \arrow[dr, "\odot_{\geqslant 0}((\bar{\jmath}_i);1)"] & \mathcal{P} \arrow[d, "\eta"] \arrow[dl, "\odot_{\geqslant 0}((\eta);1)"'] \\
\odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \mathcal{P})) \arrow[r, "\mu"'] & \odot_{\geqslant 0}((\mathcal{B}_i)_{i\in[n]}; \mathcal{P}) \arrow[r, "z"'] & \bigodot_{i=0}^n \mathcal{B}_i \odot^i_{\mathcal{A}_i} \mathcal{P}
\end{tikzcd}$$

$$
\begin{tikzcd}
((T\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant 0}((T\mathcal{A}_i)_{i\in[n]};\mathcal{P})) \arrow[r, "((\alpha_i);\odot_{\geqslant 0}((\alpha_i);1))"] \arrow[d, "((T\bar{\imath}_i);\odot_{\geqslant 0}((T\bar{\imath}_i);1))"'] \arrow[dr, phantom, "\boxed{1}"] & ((\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P})) \arrow[r, "((1)_{i\in[n]};\alpha)"] \arrow[d, "((\bar{\jmath}_i)_{i\in[n]};\odot_{\geqslant 0}((\bar{\jmath}_i)_{i\in[n]};1))"'] \arrow[dr, phantom, "\boxed{2}"] & ((\mathcal{A}_i)_{i\in[n]};\mathcal{P}) \arrow[d, "((\bar{\jmath}_i)_{i\in[n]};\eta)"] \\
((TC_i)_{i\in[n]};\odot_{\geqslant 0}((TC_i)_{i\in[n]};\mathcal{P})) \arrow[r, "((g_i);\odot_{\geqslant 0}((g_i);1))"] \arrow[d, shift right, "\phi_1"', "\wr"] \arrow[dr, phantom, "\boxed{3}"] & ((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle);\odot_{\geqslant 0}((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle);\mathcal{P})) \arrow[r, "((1)_{i\in[n]};z)"] \arrow[d, shift right, "\phi_2"', "\wr"] \arrow[dr, phantom, "\boxed{4}"] & ((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle);\mathcal{R}) \arrow[d, shift right, "\phi_3"', "\wr"] \\
\begin{array}{l}((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))\sqcup\\((T\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant 0}((T\mathcal{A}_i)_{i\in[n]};\mathcal{P}))\end{array} \arrow[r, "1\sqcup((\alpha_i);\odot_{\geqslant 0}((\alpha_i);1))"] \arrow[u, shift right, "\psi_1"'] & \begin{array}{l}((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))\sqcup\\((\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P}))\end{array} \arrow[r, "1\sqcup((1);\alpha)"] \arrow[u, shift right, "\psi_2"'] & \begin{array}{l}((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))\\\sqcup((\mathcal{A}_i)_{i\in[n]};\mathcal{P})\end{array} \arrow[u, shift right, "\psi_3"']
\end{tikzcd}
$$

satisfy the relations

$$\alpha\cdot\eta=\alpha\cdot\odot_{\geqslant 0}((\eta);1)\cdot z=\odot_{\geqslant 0}((\eta);1)\cdot\odot_{\geqslant 0}((1);\alpha)\cdot z$$
$$=\odot_{\geqslant 0}((\eta);1)\cdot\odot_{\geqslant 0}((1);\odot_{\geqslant 0}((\bar{\jmath}_i);1))\cdot\mu\cdot z=\odot_{\geqslant 0}((\bar{\jmath}_i);1)\cdot z.$$

Commutativity of squares $\boxed{3}$ and $\boxed{4}$ with the vertical arrows ψ_k is proven separately on each of summands of the source of the square. On $((\mathcal{C}_i)_{i\in[n]};\mathcal{C}_0(0))$ commutativity holds due to Lemma 1.34. On $((\mathcal{A}_i)_{i\in[n]};\odot_{\geqslant 0}((\mathcal{A}_i)_{i\in[n]};\mathcal{P}))$ verification reduces to diagram (1.19).

One easily finds out that all three columns of diagram on the preceding page compose to in_2:

$$((T\bar{\imath}_i);\odot_{\geqslant 0}((T\bar{\imath}_i);1))\cdot\phi_1=\mathrm{in}_2,\ ((\bar{\jmath}_i)_{i\in[n]};\odot_{\geqslant 0}((\bar{\jmath}_i)_{i\in[n]};1))\cdot\phi_2=\mathrm{in}_2,\ ((\bar{\jmath}_i)_{i\in[n]};\eta)\cdot\phi_3=\mathrm{in}_2\,.$$

Therefore, the exterior of this diagram drawn with isomorphisms ϕ_k is a pushout square. Hence, the pasting $\boxed{1\cup 2}$ of squares $\boxed{1}$ and $\boxed{2}$ (a diagram in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$) is a pushout square in ${}_n\mathrm{Op}_1^{\mathbf{gr}}$. However, a cone for a diagram $\mathcal{D}\to{}_n\mathrm{Op}_1^{\mathbf{dg}}$ is a colimiting cone iff its image under the forgetful functor ${}_n\mathrm{Op}_1^{\mathbf{dg}}\to{}_n\mathrm{Op}_1^{\mathbf{gr}}$ is a colimiting cone for the composition $\mathcal{D}\to{}_n\mathrm{Op}_1^{\mathbf{dg}}\to{}_n\mathrm{Op}_1^{\mathbf{gr}}$ (see Section 1.28). Thus, the pasting $\boxed{1\cup 2}$ is a pushout square in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$, and the colimit $A\langle M,\beta\rangle$ is isomorphic to $((\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle)_{i\in[n]};\bigodot_{i=0}^n\mathcal{A}_i\langle\mathcal{M}_i,\beta_i\rangle\odot_{\mathcal{A}_i}^i\mathcal{P})$ in ${}_n\mathrm{Op}_1^{\mathbf{dg}}$. $\square$

1.37. The lax $\mathcal{C}at$-multifunctor *hom*. Starting with an arbitrary braided **dg**-multicategory C we construct a lax $\mathcal{C}at$-multifunctor $hom:\mathsf{B}\to\mathsf{DG}$, where the $\mathcal{C}at$-multicategory B has $\operatorname{Ob}\mathsf{B}=\operatorname{Ob}\mathsf{C}$. Any symmetric **dg**-multicategory is obviously braided. For any sequence $(A_i)_{i\in I}$, B of objects of B the category $\mathsf{B}((A_i)_{i\in I};B)$ is the terminal category $\mathbb{1}$. An arbitrary lax $\mathcal{C}at$-multifunctor $\mathsf{B}\to\mathsf{DG}$ assigns an object of $\mathbf{dg}^{\mathbb{N}^I}$ to a sequence $(A_i)_{i\in I}$, B. In the case of *hom* this is the object $hom((A_i)_{i\in I};B)\in\operatorname{Ob}\mathbf{dg}^{\mathbb{N}^I}$ given by

$$hom((A_i)_{i\in I};B)((n^i)_{i\in I})=\mathsf{C}\big(({}^{n^i}A_i)_{i\in I};B\big).$$

A 2-morphism has to be given for each labelled $[I]$-tree $(t,\ell):[I]\to\mathcal{O}_{\mathsf{sk}}(\operatorname{Ob}\mathsf{C})$ and each t-tree $\tau:t\to\mathcal{O}_{\mathsf{sk}}$:

$$\mathrm{comp}^I_\tau:\bigotimes^{h\in I}\bigotimes^{b\in t(h)}\bigotimes^{p\in\tau(h,b)}hom\big((A^a_{h-1})_{a\in t_h^{-1}b};A^b_h\big)\big((|\tau^{-1}_{(h-1,a)\to(h,b)}(p)|)_{a\in t_h^{-1}b}\big)$$
$$=\bigotimes^{h\in I}\bigotimes^{b\in t(h)}\bigotimes^{p\in\tau(h,b)}\mathsf{C}\big(({}^{|\tau^{-1}_{(h-1,a)\to(h,b)}(p)|}A^a_{h-1})_{a\in t_h^{-1}b};A^b_h\big)$$
$$\to\mathsf{C}\big(({}^{|\tau(0,a)|}A^a_0)_{a\in t(0)};A^1_l\big)=hom\big((A^a_0)_{a\in t(0)};A^1_l\big)\big((|\tau(0,a)|)_{a\in t(0)}\big).$$

Here $0=\min[I]$, $l=\max[I]$. As such we take multiplication $\mu^{\tilde{\tau}}_{\mathsf{C}}$ in **dg**-multicategory C associated with the labelled braided tree $\tilde{\tau}$ with $\tilde{\tau}(h)=\sqcup_{b\in t(h)}\tau(h,b)$ and with the successor map

$$S_{\tilde{\tau}}=\tilde{\tau}_h:\sqcup_{a\in t(h-1)}\tau(h-1,a)\to\sqcup_{b\in t(h)}\tau(h,b)$$

induced by the map $t_h : t(h-1) \to t(h)$ of indexing sets and by the maps $\tau_{(h-1,a)\to(h,t_h(a))} : \tau(h-1,a) \to \tau(h,t_h(a))$ of summands. Label mappings $\ell : \tilde\tau(h) \to \operatorname{Ob}\mathsf{C}$ associate A_h^b to any $p \in \tau(h,b)$. Equations (1.11) follow from [BLM08, equation (2.25.1)] written for the algebra C in the lax Monoidal category ${}^{\operatorname{Ob}\mathsf{C}}\mathcal{PMQ}_{\mathbf{dg}}$. The procedure of forming $\tilde\tau$ commutes with restricting trees along maps $\psi : [J] \to [I]$.

An example of such symmetric **dg**-multicategory C comes from $\mathsf{C}_{\Bbbk}$ – the closed symmetric multicategory of complexes of $\Bbbk$-modules and their chain maps [BLM08, Example 3.18]. It is representable by the symmetric Monoidal category **dg** of complexes and chain maps [BLM08, Example 3.27]. We take for C the associated enriched symmetric multicategory $\underline{\mathsf{C}}_{\Bbbk}$, which is a $\mathsf{C}_{\Bbbk}$-multicategory, or equivalently, a **dg**-multicategory. The composition in $\underline{\mathsf{C}}_{\Bbbk}$ has a natural meaning: this is a composition (of tensor products) of homogeneous maps, taking into account the Koszul rule.

The $\mathcal{C}at$-multicategory B from Section 1.37 is also a $\mathcal{C}at$-span multicategory with the discrete category ${}_{\mathsf{t}}\mathsf{B}$, $\operatorname{Ob}{}_{\mathsf{t}}\mathsf{B} = \operatorname{Ob}\mathsf{C}$. The lax $\mathcal{C}at$-multifunctor $hom : \mathsf{B} \to \mathsf{DG}$ is simultaneously a lax $\mathcal{C}at$-span multifunctor $hom : \mathsf{B} \to \mathsf{DG}$. The prism equation from Definition 1.11 takes in notation of Section 0.27.1 the form

$$\begin{aligned}
\big[&\circledast_{\mathsf{DG}}(t)\big(hom((A_{h-1}^a)_{a\in t_h^{-1}b}; A_h^b)\big)_{(h,b)\in\mathrm{v}(t)} \\
&\xrightarrow{\lambda^f_{\mathsf{DG}}} \circledast_{\mathsf{DG}}(t_\psi)\big(\circledast_{\mathsf{DG}}(t|^c_{[\psi(g-1),\psi(g)]})(hom((A_{h-1}^a)_{a\in t_h^{-1}b}; A_h^b))_{(h,b)\in\mathrm{v}(t|^c_{[\psi(g-1),\psi(g)]})}\big)_{(g,c)\in\mathrm{v}(t_\psi)} \\
&\qquad\xrightarrow{\circledast_{\mathsf{DG}}(t_\psi)\operatorname{comp}(t|^c_{[\psi(g-1),\psi(g)]})} \circledast_{\mathsf{DG}}(t_\psi)\big(hom((A_{\psi(g-1)}^a)_{a\in t_{\psi,g}^{-1}c}; A_{\psi g}^c)\big)_{(g,c)\in\mathrm{v}(t_\psi)} \\
&\qquad\qquad\xrightarrow{\operatorname{comp}(t_\psi)} hom((A_0^a)_{a\in t(0)}; A^1_{\max[I]})\big] = \operatorname{comp}(t). \qquad (1.20)
\end{aligned}$$

Let us discuss the relationship between the suspension and the composition $\mu^T_{\underline{\mathsf{C}}_{\Bbbk}}$ for a symmetric free $T : [l] \to \mathcal{S}_{\mathsf{sk}}$.

Let $g : U \to W$, $f_i : X_i \to Y_i$, $1 \leqslant i \leqslant k$ be homogeneous maps of certain degrees. For any $1 \leqslant j \leqslant k$ the maps

$$\underline{\mathsf{C}}_{\Bbbk}\big((1)_{i<j}, f_j, (1)_{i>j}; 1\big) : \underline{\mathsf{C}}_{\Bbbk}\big((X_i)_{i<j}, (Y_i)_{i\geqslant j}; U\big) \to \underline{\mathsf{C}}_{\Bbbk}\big((X_i)_{i\leqslant j}, (Y_i)_{i>j}; U\big)$$

are defined as the precomposition with f_j, $h \mapsto (-1)^{h\cdot f_j}(1^{j-1} \times f_j \times 1^{k-j}) \cdot h$. The map

$$\underline{\mathsf{C}}_{\Bbbk}\big((1)_{i=1}^k; g\big) : \underline{\mathsf{C}}_{\Bbbk}\big((X_i)_{i=1}^k; U\big) \to \underline{\mathsf{C}}_{\Bbbk}\big((X_i)_{i=1}^k; W\big)$$

is defined as the postcomposition with g, $h \mapsto h \cdot g$. By convention, the map

$$\underline{\mathsf{C}}_{\Bbbk}\big(f_1, f_2, \dots, f_k; g\big) : \underline{\mathsf{C}}_{\Bbbk}\big((Y_i)_{i=1}^k; U\big) \to \underline{\mathsf{C}}_{\Bbbk}\big((X_i)_{i=1}^k; W\big)$$

is the composition (in this order)

$$\underline{\mathsf{C}}_{\Bbbk}\big(f_1, 1, \dots, 1; 1\big) \cdot \underline{\mathsf{C}}_{\Bbbk}\big(1, f_2, \dots, 1; 1\big) \cdot \dots \cdot \underline{\mathsf{C}}_{\Bbbk}\big(1, \dots, 1, f_k; 1\big) \cdot \underline{\mathsf{C}}_{\Bbbk}\big(1, \dots, 1, 1; g\big).$$

Factors of this product commute up to the sign depending on parity of the product of degrees of factors.

1.38 Lemma. *For arbitrary complexes $A^b_h \in \operatorname{Ob} \underline{\mathsf{C}}_{\Bbbk}$ the following square commutes up to the sign $(-1)^{c(T)}$*

$$
\begin{tikzcd}
\bigotimes^{h\in I} \bigotimes^{b\in T(h)} \underline{\mathsf{C}}_{\Bbbk}\bigl((sA^a_{h-1})_{a\in T_h^{-1}b}; sA^b_h\bigr) \arrow[r, "\mu^T_{\underline{\mathsf{C}}_{\Bbbk}}"] \arrow[d, "\otimes^{h\in I}\otimes^{b\in T(h)}\underline{\mathsf{C}}_{\Bbbk}({}^{T_h^{-1}b}\sigma;\sigma^{-1})"'] \arrow[dr, phantom, "(-1)^{c(T)}"] & \underline{\mathsf{C}}_{\Bbbk}\bigl((sA^a_0)_{a\in T(0)}; sA^1_l\bigr) \arrow[d, "\underline{\mathsf{C}}_{\Bbbk}({}^{T(0)}\sigma;\sigma^{-1})"] \\
\bigotimes_{h\in I} \bigotimes_{b\in T(h)} \underline{\mathsf{C}}_{\Bbbk}\bigl((A^a_{h-1})_{a\in T_h^{-1}b}; A^b_h\bigr) \arrow[r, "\mu^T_{\underline{\mathsf{C}}_{\Bbbk}}"'] & \underline{\mathsf{C}}_{\Bbbk}\bigl((A^a_0)_{a\in T(0)}; A^1_l\bigr)
\end{tikzcd}
$$

where

$$
c(T) = \sum_{h=1}^{l-1} \sum_{b=1}^{|T(h)|} (b-1)(1-|T_h^{-1}b|) + \sum_{h=1}^{l-1} |\{(x,y) \in T(h-1)^2 \mid x<y,\ T_h(x) > T_h(y)\}|.
$$

Proof. The sign coincides with the sign of permutation of the expression $\otimes^{h\in I} \otimes^{b\in T(h)} (\sigma^{\otimes T_h^{-1}b} \otimes \sigma^{-1})$, followed by cancellation of matching σ^{-1} and σ, resulting in $(-1)^{c(T)}\sigma^{\otimes T(0)} \otimes \sigma^{-1}$. This permutation can be performed in two steps applied to each floor of the tree starting from the root. At the first step factors of $\sigma^{\otimes T_h^{-1}b}$ (starting from the right) are moved to the left towards the matching σ^{-1}. This explains appearance of the first sum in $c(T)$. At the second step factors $\otimes^{b\in T(h)}\sigma^{\otimes T_h^{-1}b}$ are permuted to $\sigma^{\otimes T(h-1)}$ accordingly to the map $T_h : T(h-1) \to T(h)$. This is reflected by the second sum in $c(T)$. □

Let $g : U \to W$, $f_i : X_i \to Y_i$, $1 \leqslant i \leqslant k$ be homogeneous maps of certain degrees. Then

$$
\mathit{hom}\bigl((f_i)_{i\in I}; g\bigr) : \mathit{hom}\bigl((Y_i)_{i\in I}; U\bigr) \to \mathit{hom}\bigl((X_i)_{i\in I}; W\bigr)
$$

denotes the collection of homogeneous maps

$$
\mathit{hom}\bigl((f_i)_{i\in I}; g\bigr) = \underline{\mathsf{C}}_{\Bbbk}\bigl(({}^{n^i}f_i)_{i\in I}; g\bigr) : \underline{\mathsf{C}}_{\Bbbk}\bigl(({}^{n^i}Y_i)_{i\in I}; U\bigr) \to \underline{\mathsf{C}}_{\Bbbk}\bigl(({}^{n^i}X_i)_{i\in I}; W\bigr).
$$

1.39 Corollary. *For each t-tree τ as above the following square commutes up to the sign $(-1)^{c(\tilde{\tau})}$*

$$
\begin{tikzcd}[column sep=small]
\bigotimes^{h\in I} \bigotimes^{b\in t(h)} \bigotimes^{p\in\tau(h,b)} \mathit{hom}\bigl((sA^a_{h-1})_{a\in t_h^{-1}b}; sA^b_h\bigr)\bigl((|\tau^{-1}_{(h-1,a)}(p)|)_{a\in t_h^{-1}b}\bigr) \arrow[dr, "\mathrm{comp}^I_\tau"] \arrow[dd, "\otimes^{h\in I}\otimes^{b\in t(h)}\otimes^{p\in\tau(h,b)} \mathit{hom}({}^{t_h^{-1}b}\sigma;\sigma^{-1})((|\tau^{-1}_{(h-1,a)}(p)|)_{a\in t_h^{-1}b})"'] & \\
& \mathit{hom}\bigl((sA^a_0)_{a\in t(0)}; sA^1_l\bigr)\bigl((|\tau(0,a)|)_{a\in t(0)}\bigr) \arrow[dd, "\mathit{hom}({}^{t(0)}\sigma;\sigma^{-1})((|\tau(0,a)|)_{a\in t(0)})"] \\
\bigotimes_{h\in I} \bigotimes_{b\in t(h)} \bigotimes_{p\in\tau(h,b)} \mathit{hom}\bigl((A^a_{h-1})_{a\in t_h^{-1}b}; A^b_h\bigr)\bigl((|\tau^{-1}_{(h-1,a)}(p)|)_{a\in t_h^{-1}b}\bigr) \arrow[dr, "\mathrm{comp}^I_\tau"'] \arrow[ur, phantom, "(-1)^{c(\tilde{\tau})}"] & \\
& \mathit{hom}\bigl((A^a_0)_{a\in t(0)}; A^1_l\bigr)\bigl((|\tau(0,a)|)_{a\in t(0)}\bigr)
\end{tikzcd}
$$

where $t : [l] \to \mathcal{O}_{\mathsf{sk}}$, $\tau : t \to \mathcal{O}_{\mathsf{sk}}$,

$$
\begin{aligned}
c(\tilde{\tau}) = \sum_{h=1}^{l-1} \sum_{x \in \tilde{\tau}(h)} (1 - |\tilde{\tau}_h^{-1} x|) |\{y \in \tilde{\tau}(h) \mid y < x\}| \\
+ \sum_{h=1}^{l-1} \sum_{\substack{a,b \in t(h-1) \\ a<b,\, t_h a = t_h b}} |\{(u,v) \in \tau(h-1,a) \times \tau(h-1,b) \mid \tau_{(h-1,a)} u > \tau_{(h-1,b)} v\}|. \quad (1.21)
\end{aligned}
$$

Proof. Let $a, b \in t(h-1)$, $u \in \tau(h-1,a)$, $v \in \tau(h-1,b)$. Then $x = (a,u)$, $y = (b,v) \in \tilde{\tau}(h)$. Assume that $x < y$. Several cases occur. If $a = b$, $u < v$, then $\tilde{\tau}_h(x) \leqslant \tilde{\tau}_h(y)$. If $a < b$, $t_h a < t_h b$, then $\tilde{\tau}_h(x) < \tilde{\tau}_h(y)$. If $a < b$, $t_h a = t_h b$, then $\tilde{\tau}_h(x) > \tilde{\tau}_h(y) \iff \tau_{(h-1,a)} u > \tau_{(h-1,b)} v$. □

1.40. Multicategory of operad modules. Note that $n \wedge 1$-operad modules form a $\mathcal{C}at$-span multiquiver M with ${}_{\mathsf{t}}\mathsf{M} = \mathrm{Op}$, the category of operads, with the functors

$$
(\mathcal{A}_i)_{i \in I} \xleftarrow{\text{src}} ((\mathcal{A}_i)_{i \in I}; \mathcal{P}; \mathcal{B}) \xmapsto{\text{tgt}} \mathcal{B}.
$$

The $\mathcal{C}at$-span multiquiver M becomes a weak $\mathcal{C}at$-span multicategory when equipped with the tensor product $\circledast_{\mathsf{M}}$ defined as follows. Consider a tree $t : [p] \to \mathcal{O}_{\mathsf{sk}}$ like in (0.11). If $p = 0$ the morphism $\circledast^{\varnothing} : \boxdot^{\varnothing}\mathsf{M} = {}_{\mathsf{t}}\mathsf{M} \to \mathsf{M}$ takes an operad $\mathcal{A}$ to the regular $\mathcal{A}$-bimodule $(\mathcal{A}; \mathcal{A}; \mathcal{A})$. If $p = 1$ the morphism $\circledast^{\mathbf{1}} : \boxdot^{\mathbf{1}}\mathsf{M} \to \mathsf{M}$ is the natural isomorphism. If $p > 1$, $g \in \mathbb{N}$, $0 < g < p$, introduce a new tree by doubling the g-th level of t:

$$
t^g_+ = \bigl(t(0) \xrightarrow{t_1} \cdots \to t(g-1) \xrightarrow{t_g} t(g) \xrightarrow{\text{id}} t(g) \xrightarrow{t_{g+1}} t(g+1) \to \dots \xrightarrow{t_p} t(p) = \mathbf{1}\bigr).
$$

Consider a diagram of the form

$$
\begin{tikzcd}
 & \bullet \arrow[d, shift left] \arrow[d, shift right] & \\
\bullet \arrow[r, shift left] \arrow[r, shift right] & \blacklozenge & \bullet \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}
\quad \cdots \quad (1.22)
$$

where $p-1$ pairs of parallel arrows are

$$
\bullet = \circledast_{\mathsf{DG}}(t^g_+)\bigl((\mathcal{P}^b_h)_{(h,b)\in \mathrm{v}(t)}, (\mathcal{A}^c_g)_{c \in t(g)}\bigr) \rightrightarrows \circledast_{\mathsf{DG}}(t)(\mathcal{P}^b_h)_{(h,b)\in \mathrm{v}(t)} = \blacklozenge \qquad (1.23)
$$

for $0 < g < p$. The first arrow corresponds to simultaneous right actions ρ of $\mathcal{A}^c_g$ on $\mathcal{P}^c_g$ and the second arrow consists of simultaneous left actions λ of $(\mathcal{A}^c_g)_{c \in t^{-1}_{g+1} b}$ on $\mathcal{P}^b_{g+1}$, $b \in t(g+1)$. In detail: we take the two embeddings $\psi : [p] \to [p+1]$ missing g or $g+1$, then $(t^g_+)_\psi = t$. We use isomorphism (1.10)

$$
\begin{aligned}
\circledast_{\mathsf{DG}}(t^g_+)\bigl((\mathcal{P}^b_h)_{(h,b)\in \mathrm{v}(t)}, (\mathcal{A}^c_g)_{c \in t(g)}\bigr) \xrightarrow{\sim} \circledast_{\mathsf{DG}}(t)\bigl((\mathcal{P}^b_h)^{b \in t(h)}_{h<g}, (X_b)_{b \in t(g)}, (\mathcal{P}^b_h)^{b \in t(h)}_{h>g}\bigr) \\
\longrightarrow \circledast_{\mathsf{DG}}(t)(\mathcal{P}^b_h)_{(h,b)\in \mathrm{v}(t)}
\end{aligned}
$$

where X_b stands for $\circledast_{\mathsf{DG}}(t_g^{-1}b \to \{b\} \to \{b\})(\mathcal{P}_g^b, \mathcal{A}_g^b)$ (resp. for $\circledast_{\mathsf{DG}}(t_g^{-1}b \xrightarrow{\mathrm{id}} t_g^{-1}b \to \{b\})((\mathcal{A}_g^c)_{c\in t_{g+1}^{-1}b}, \mathcal{P}_{g+1}^b))$, mapped by ρ (resp. λ) to $\mathcal{P}_g^b$ (resp. $\mathcal{P}_{g+1}^b$). The summand of the source of (1.23) indexed by $\tau_+^g : t_+^g \to \mathcal{O}_{\mathsf{sk}}$ is mapped by the first (resp. the second) arrow to the summand indexed by t-tree τ_ρ^g (resp. τ_λ^g), given by $\bigl(t \xrightarrow{\psi-} t_+^g \xrightarrow{\tau_+^g} \mathcal{O}_{\mathsf{sk}}\bigr)$, where the functor $\psi- : t \to t_+^g$ corresponds to the embedding ψ of vertices. Thus,

$$\pi : \circledast_{\mathsf{DG}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)} \to \circledast_{\mathsf{M}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)} \tag{1.24}$$

determines the colimiting cone of (1.22).

Another definition is based on diagram (1.22) with pairs of arrows

$$\circledast_{\mathsf{DG}}(t_+^g)((\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}, \mathcal{A}_g^c) \rightrightarrows \circledast_{\mathsf{DG}}\,(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}$$

indexed by elements (g,c) of $\mathrm{v}'(t) = \mathrm{v}(t) - \{\mathrm{root}\}$ – the set of internal edges of t (edges from t_1 are excluded). This pair is obtained from pair (1.23) by precomposing with units $\eta : \Bbbk \to \mathcal{A}_g^b$ for $b \in t(g) - \{c\}$ ($\Bbbk$ being the unit operad). The operad $\mathcal{A}_g^c$ still acts via ρ on $\mathcal{P}_g^c$ on the right and via λ^c on $\mathcal{P}_{g+1}^{t_{g+1}c}$ on the left. The colimit of (1.22) is $\circledast_{\mathsf{M}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}$. Equivalence of the two definitions follows from the possibility to take elements of the operads equal to the unity for all operads $\mathcal{A}_g^b$ but one.

Notice that for $p > 0$ the collection $\bigl((\mathcal{A}_0^b)_{b\in t(0)}; \circledast_{\mathsf{DG}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}; \mathcal{A}_p^1\bigr)$ has an operad polymodule structure. The right action is constructed with the help of two increasing injections: $\psi_1 : [2] \to [p+1]$, $0 \mapsto 0$, $1 \mapsto p$, $2 \mapsto p+1$, and $\psi_2 : [p] \to [p+1]$ missing p. Namely,

$$\begin{aligned}\rho = \bigl[\bigl(\circledast_{\mathsf{DG}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}\bigr) \odot_0 \mathcal{A}_p^1 \xrightarrow[\sim]{\lambda_{\psi_1}^{-1}} \circledast_{\mathsf{DG}}\,(t_+^p)((\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}, \mathcal{A}_p^1)\\ \xrightarrow[\sim]{\lambda_{\psi_2}} \circledast_{\mathsf{DG}}\,(t)((\mathcal{P}_h^b)_{1\leqslant h<p}^{b\in t(h)}, \mathcal{P}_p^1 \odot_0 \mathcal{A}_p^1) \xrightarrow{\circledast_{\mathsf{DG}}(t)((1),\rho)} \circledast_{\mathsf{DG}}\,(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}\bigr],\end{aligned} \tag{1.25}$$

where $\mathcal{P}_p^1 \odot_0 \mathcal{A}_p^1 = \circledast_{\mathsf{DG}}(t(p-1) \to \mathbf{1} \to \mathbf{1})(\mathcal{P}_p^1, \mathcal{A}_p^1)$. The left action uses the following increasing injections: $\psi_1 : [2] \to [p+1]$, $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto p+1$, and $\psi_2 : [p] \to [p+1]$ missing 1. Namely,

$$\begin{aligned}\lambda = \bigl[\odot_{>0}\bigl((\mathcal{A}_0^b)_{b\in t(0)}; \circledast_{\mathsf{DG}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}\bigr) \xrightarrow[\sim]{\lambda_{\psi_1}^{-1}} \circledast_{\mathsf{DG}}\,(t_+^0)((\mathcal{A}_0^b)_{b\in t(0)}; (\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}) \xrightarrow[\sim]{\lambda_{\psi_2}}\\ \circledast_{\mathsf{DG}}\,(t)\bigl[\bigl(\odot_{>0}((\mathcal{A}_0^c)_{c\in t_1^{-1}b}; \mathcal{P}_1^b)\bigr)_{b\in t(1)}, (\mathcal{P}_h^b)_{1<h\leqslant p}^{b\in t(h)}\bigr] \xrightarrow{\circledast_{\mathsf{DG}}(t)((\lambda),1)} \circledast_{\mathsf{DG}}\,(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}\bigr],\end{aligned} \tag{1.26}$$

where $\odot_{>0}((\mathcal{A}_0^c)_{c\in t_1^{-1}b}; \mathcal{P}_1^b) = \circledast_{\mathsf{DG}}(t_1^{-1}b \xrightarrow{\mathrm{id}} t_1^{-1}b \to \{b\})((\mathcal{A}_0^c)_{c\in t_1^{-1}b}; \mathcal{P}_1^b)$. These (outer) actions commute with inner actions of $\mathcal{A}_g^c$, thus they project to the quotient $\circledast_{\mathsf{M}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}$, making $\bigl((\mathcal{A}_0^b)_{b\in t(0)}; \circledast_{\mathsf{M}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}; \mathcal{A}_p^1\bigr)$ into an operad polymodule.

There is a lax $\mathcal{C}at$-span multifunctor $\mathsf{M} \to \mathsf{DG}$, $((\mathcal{A}_i)_{i\in I}; \mathcal{P}; \mathcal{B}) \mapsto \mathcal{P}$. The component $\pi : \circledast_{\mathsf{DG}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)} \to \circledast_{\mathsf{M}}(t)(\mathcal{P}_h^b)_{(h,b)\in \mathrm{v}(t)}$ is that of definition (1.24). We lift *hom* to a

lax $\mathcal{C}at$-span multifunctor $\mathcal{H}om : \mathsf{B} \to \mathsf{M}$ so that $\bigl(\mathsf{B} \xrightarrow{\mathcal{H}om} \mathsf{M} \longrightarrow \mathsf{DG}\bigr) = hom$. Namely, ${}_{\mathsf{t}}\mathcal{H}om : {}_{\mathsf{t}}\mathsf{B} \to {}_{\mathsf{t}}\mathsf{M}$, $B \mapsto \mathcal{E}nd\,B$, and $\mathcal{H}om : \mathsf{B} \to \mathsf{M}$,

$$((A_i)_{i\in I}; B) \mapsto ((\mathcal{E}nd\,A_i)_{i\in I}; hom((A_i)_{i\in I}; B); \mathcal{E}nd\,B) = \mathcal{H}om((A_i)_{i\in I}; B).$$

Prism equation (1.20) projects to

$$\begin{aligned}
&\bigl[\circledast_{\mathsf M}(t)\bigl(\mathcal{H}om((A^a_{h-1})_{a\in t_h^{-1}b}; A^b_h)\bigr)_{(h,b)\in \mathrm{v}(t)} \\
&\xrightarrow{\lambda^f_{\mathsf M}} \circledast_{\mathsf M}(t_\psi)\bigl(\circledast_{\mathsf M}(t|^c_{[\psi(g-1),\psi(g)]})(\mathcal{H}om((A^a_{h-1})_{a\in t_h^{-1}b}; A^b_h))_{(h,b)\in\mathrm{v}(t|^c_{[\psi(g-1),\psi(g)]})}\bigr)_{(g,c)\in\mathrm{v}(t_\psi)} \\
&\qquad\xrightarrow{\circledast_{\mathsf M}(t_\psi)\,\mathrm{comp}(t|^c_{[\psi(g-1),\psi(g)]})} \circledast_{\mathsf M}(t_\psi)\bigl(\mathcal{H}om((A^a_{\psi(g-1)})_{a\in t^{-1}_{\psi,g}c}; A^c_{\psi g})\bigr)_{(g,c)\in\mathrm{v}(t_\psi)} \\
&\qquad\qquad\xrightarrow{\mathrm{comp}(t_\psi)} \mathcal{H}om((A^a_0)_{a\in t(0)}; A^1_{\max[I]})\bigr] = \mathrm{comp}(t).
\end{aligned} \tag{1.27}$$

1.41 Example. Let us consider in detail the particular case of $p = 2$. Then $g = 1$, $t = \{\sqcup_{c=1}^n \mathbf{l}^c \to \mathbf{n} \to \mathbf{1}\}$, $t^1_+ = \{\sqcup_{c=1}^n \mathbf{l}^c \to \mathbf{n} \xrightarrow{\mathrm{id}} \mathbf{n} \to \mathbf{1}\}$. An arbitrary tree $\tau^1_+ : t^1_+ \to \mathcal{O}_{\mathsf{sk}}$ has the form

$$\tau^1_+ = \begin{tikzcd}
\sqcup_{p=1}^{u^n}\mathbf{r}_p^{n,l^n} \arrow[ddr] & & & \\
\cdots & & & \\
\sqcup_{p=1}^{u^n}\mathbf{r}_p^{n,1} \arrow[r] & \mathbf{u}^n = \sqcup_{q=1}^{k^n}\mathbf{j}_q^n \arrow[r] & \mathbf{k}^n \arrow[dddr] & \\
\cdots & \cdots & & \\
\sqcup_{p=1}^{u^c}\mathbf{r}_p^{c,l^c} \arrow[dr] & \cdots \arrow[r] & \cdots \arrow[dr] & \\
\cdots & \mathbf{u}^c = \sqcup_{q=1}^{k^c}\mathbf{j}_q^c \arrow[r] & \mathbf{k}^c \arrow[r] & \mathbf{1} \\
\sqcup_{p=1}^{u^c}\mathbf{r}_p^{c,1} \arrow[ur] & \cdots \arrow[r] & \cdots \arrow[ur] & \\
\cdots & \cdots & & \\
\sqcup_{p=1}^{u^1}\mathbf{r}_p^{1,l^1} \arrow[r] & \mathbf{u}^1 = \sqcup_{q=1}^{k^1}\mathbf{j}_q^1 \arrow[r] & \mathbf{k}^1 \arrow[uuur] & \\
\cdots & & & \\
\sqcup_{p=1}^{u^1}\mathbf{r}_p^{1,1} \arrow[uur] & & &
\end{tikzcd} . \tag{1.28}$$

The resulting t-trees τ^1_λ and τ^1_ρ are presented below:

$$\tau^1_\lambda = \begin{tikzcd}
\sqcup_{p=1}^{u^n}\mathbf{r}_p^{n,l^n} \arrow[ddr] & & \\
\cdots & & \\
\sqcup_{p=1}^{u^n}\mathbf{r}_p^{n,1} \arrow[r] & \mathbf{u}^n \arrow[dddr] & \\
\cdots & \cdots & \\
\sqcup_{p=1}^{u^c}\mathbf{r}_p^{c,l^c} \arrow[dr] & \cdots \arrow[dr] & \\
\cdots & \mathbf{u}^c \arrow[r] & \mathbf{1} \\
\sqcup_{p=1}^{u^c}\mathbf{r}_p^{c,1} \arrow[ur] & \cdots \arrow[ur] & \\
\cdots & \cdots & \\
\sqcup_{p=1}^{u^1}\mathbf{r}_p^{1,l^1} \arrow[r] & \mathbf{u}^1 \arrow[uuur] & \\
\cdots & & \\
\sqcup_{p=1}^{u^1}\mathbf{r}_p^{1,1} \arrow[uur] & &
\end{tikzcd} , \quad
\tau^1_\rho = \begin{tikzcd}
\sqcup_{q=1}^{k^n}\mathbf{s}_q^{n,l^n} \arrow[ddr] & & \\
\cdots & & \\
\sqcup_{q=1}^{k^n}\mathbf{s}_q^{n,1} \arrow[r] & \mathbf{k}^n \arrow[dddr] & \\
\cdots & \cdots & \\
\sqcup_{q=1}^{k^c}\mathbf{s}_q^{c,l^c} \arrow[dr] & \cdots \arrow[dr] & \\
\cdots & \mathbf{k}^c \arrow[r] & \mathbf{1} \\
\sqcup_{q=1}^{k^c}\mathbf{s}_q^{c,1} \arrow[ur] & \cdots \arrow[ur] & \\
\cdots & \cdots & \\
\sqcup_{q=1}^{k^1}\mathbf{s}_q^{1,l^1} \arrow[r] & \mathbf{k}^1 \arrow[uuur] & \\
\cdots & & \\
\sqcup_{q=1}^{k^1}\mathbf{s}_q^{1,1} \arrow[uur] & &
\end{tikzcd} . \tag{1.29}$$

In the latter case we use the notation $r^{b,a}_{q,v} = r^{b,a}_p$ for $q \in \mathbf{k}^b$, $v \in \mathbf{j}^b_q$, where $p = j^b_1 + \cdots + j^b_{q-1} + v \in \mathbf{u}^b$. Furthermore,

$$s^{b,a}_q = \sum_{v=1}^{j^b_q} r^{b,a}_{q,v} = \sum_{p=j^b_1+\cdots+j^b_{q-1}+1}^{j^b_1+\cdots+j^b_{q-1}+j^b_q} r^{b,a}_p .$$

Direct summands corresponding to these trees are related by the two maps. The second arrow is

$$
\begin{aligned}
\circledast_{\mathsf{G}}(t^1_+)((\mathcal{P}^b)_{b\in\mathbf{n}},(\mathcal{B}^b)_{b\in\mathbf{n}},\mathcal{Q}(\tau^1_+) &= \Big[\bigotimes^{b\in\mathbf{n}}\bigotimes^{p\in\mathbf{u}^b}\mathcal{P}^b((r^{b,a}_p)_{a\in\mathbf{l}^b})\Big]\otimes\Big(\bigotimes^{b\in\mathbf{n}}\bigotimes^{q\in\mathbf{k}^b}\mathcal{B}^b(j^b_q)\Big)\otimes\mathcal{Q}((k^b)_{b\in\mathbf{n}}) \\
&\xrightarrow{1\otimes\lambda}\Big[\bigotimes^{b\in\mathbf{n}}\bigotimes^{p\in\mathbf{u}^b}\mathcal{P}^b((r^{b,a}_p)_{a\in\mathbf{l}^b})\Big]\otimes\mathcal{Q}((u^b)_{b\in\mathbf{n}}) = \circledast_{\mathsf{DG}}(t)((\mathcal{P}^b)_{b\in\mathbf{n}},\mathcal{Q})(\tau^1_\lambda).
\end{aligned} \tag{1.30}
$$

The first arrow includes isomorphism (1.10):

$$
\begin{aligned}
&\Big[\bigotimes^{b\in\mathbf{n}}\bigotimes^{p\in\mathbf{u}^b}\mathcal{P}^b((r^{b,a}_p)_{a\in\mathbf{l}^b})\Big]\otimes\Big(\bigotimes^{b\in\mathbf{n}}\bigotimes^{q\in\mathbf{k}^b}\mathcal{B}^b(j^b_q)\Big)\otimes\mathcal{Q}((k^b)_{b\in\mathbf{n}})\xrightarrow{\sim} \\
&\quad\Big\{\bigotimes^{b\in\mathbf{n}}\bigotimes^{q\in\mathbf{k}^b}\Big[\Big(\bigotimes^{v\in\mathbf{j}^b_q}\mathcal{P}^b((r^{b,a}_{q,v})_{a\in\mathbf{l}^b})\otimes\mathcal{B}^b(j^b_q)\Big]\Big\}\otimes\mathcal{Q}((k^b)_{b\in\mathbf{n}})\xrightarrow{(\otimes^{b\in\mathbf{n}}\otimes^{q\in\mathbf{k}^b}\rho)\otimes1} \\
&\qquad\Big[\bigotimes^{b\in\mathbf{n}}\bigotimes^{q\in\mathbf{k}^b}\mathcal{P}^b((s^{b,a}_q)_{a\in\mathbf{l}^b})\Big]\otimes\mathcal{Q}((k^b)_{b\in\mathbf{n}}) = \circledast_{\mathsf{DG}}(t)((\mathcal{P}^b)_{b\in\mathbf{n}},\mathcal{Q})(\tau^1_\rho).
\end{aligned} \tag{1.31}
$$

In the special case with distinguished $c\in\mathbf{n}$ we take $j^b_q=1$ for $b\neq c$, $q\in\mathbf{k}^b$. The above maps are precomposed with the units $\eta:\Bbbk\to\mathcal{B}^b(1)$ for $b\neq c$. For $b=c$ the action ρ of $\mathcal{B}^c$ on $\mathcal{P}^c$ is the essential part of the first arrow, and the action λ^c of $\mathcal{B}^c$ on $\mathcal{Q}$ is left in the second arrow.

For $t=\{\sqcup^n_{c=1}\mathbf{l}^c\to\mathbf{n}\to\mathbf{1}\}$ there is $t^2_+=\{\sqcup^n_{c=1}\mathbf{l}^c\to\mathbf{n}\to\mathbf{1}\to\mathbf{1}\}$. A t^2_+-tree has the form

$$
\tau^2_+ = \quad
\begin{array}{ccccc}
\sqcup^{u^n}_{p=1}\mathbf{r}^{n,l^n}_p \searrow & & & & \\
\cdots & & & & \\
\sqcup^{u^n}_{p=1}\mathbf{r}^{n,1}_p \longrightarrow & \mathbf{u}^n=\sqcup^w_{v=1}\mathbf{u}^n_v \searrow & & & \\
\cdots & \cdots & & & \\
\sqcup^{u^c}_{p=1}\mathbf{r}^{c,l^c}_p \searrow & \cdots & & & \\
\cdots & \mathbf{u}^c=\sqcup^w_{v=1}\mathbf{u}^c_v \longrightarrow & \mathbf{w} & \longrightarrow & \mathbf{1} \\
\sqcup^{u^c}_{p=1}\mathbf{r}^{c,1}_p \nearrow & \cdots & & & \\
\cdots & \cdots & & & \\
\sqcup^{u^1}_{p=1}\mathbf{r}^{1,l^1}_p \longrightarrow & \mathbf{u}^1=\sqcup^w_{v=1}\mathbf{u}^1_v \nearrow & & & \\
\cdots & & & & \\
\sqcup^{u^1}_{p=1}\mathbf{r}^{1,1}_p \nearrow & & & &
\end{array} \quad .
$$

Respectively, right action (1.25) of $\mathcal{C}$ on $\circledast_{\mathsf{G}}(t)((\mathcal{P}^c)_{c\in\mathbf{n}},\mathcal{Q})$ is given on the direct summand corresponding to τ^2_+ by

$$
\begin{aligned}
&\Big[\Big(\circledast_{\mathsf{G}}(t)((\mathcal{P}^c)_{c\in\mathbf{n}},\mathcal{Q})\Big)\odot_0\mathcal{C}\Big](\tau^2_+) = \Bigg[\bigotimes^{v\in\mathbf{w}}\Bigg(\Big[\bigotimes^{c\in\mathbf{n}}\bigotimes^{u^c_1+\dots+u^c_{v-1}+u^c_v}_{p=u^c_1+\dots+u^c_{v-1}+1}\mathcal{P}^c((r^{c,g}_p)_{g\in\mathbf{l}^c})\Big]\otimes\mathcal{Q}(u_v)\Bigg)\Bigg]\otimes\mathcal{C}(w) \\
&\xrightarrow{\sim}\Big[\bigotimes^{c\in\mathbf{n}}\bigotimes^{p\in\mathbf{u}^c}\mathcal{P}^c((r^{c,g}_p)_{g\in\mathbf{l}^c})\Big]\otimes\Big[\bigotimes^{v\in\mathbf{w}}\mathcal{Q}(u_v)\Big]\otimes\mathcal{C}(w)\xrightarrow{1\otimes\rho}\Big[\bigotimes^{c\in\mathbf{n}}\bigotimes^{p\in\mathbf{u}^c}\mathcal{P}^c((r^{c,g}_p)_{g\in\mathbf{l}^c})\Big]\otimes\mathcal{Q}(u).
\end{aligned}
$$

There is also $t^0_+ = \{\sqcup_{c=1}^n \mathbf{l}^c \xrightarrow{\mathrm{id}} \sqcup_{c=1}^n \mathbf{l}^c \to \mathbf{n} \to \mathbf{1}\}$. A t^0_+-tree has the form

$$\tau^0_+ = \begin{array}{ccccccc}
\sqcup_{q=1}^{k^n} \sqcup_{\ddot{\imath}=1}^{s_q^{n,l^n}} \boldsymbol{\nu}_{q,\ddot{\imath}}^{n,l^n} & \longrightarrow & \mathbf{j}^{n,l^n} & = & \sqcup_{q=1}^{k^n} \mathbf{s}_q^{n,l^n} & \searrow & \\
 & \dots & & & \dots & & \\
\sqcup_{q=1}^{k^n} \sqcup_{\ddot{\imath}=1}^{s_q^{n,1}} \boldsymbol{\nu}_{q,\ddot{\imath}}^{n,1} & \longrightarrow & \mathbf{j}^{n,1} & = & \sqcup_{q=1}^{k^n} \mathbf{s}_q^{n,1} & \longrightarrow & \mathbf{k}^n \\
 & \dots & & & \dots & \dots & \\
\sqcup_{q=1}^{k^c} \sqcup_{\ddot{\imath}=1}^{s_q^{c,l^c}} \boldsymbol{\nu}_{q,\ddot{\imath}}^{c,l^c} & \longrightarrow & \mathbf{j}^{c,l^c} & = & \sqcup_{q=1}^{k^c} \mathbf{s}_q^{c,l^c} & \searrow & \dots \\
 & \dots & & & \dots & & \mathbf{k}^c \longrightarrow \mathbf{1}\ . \\
\sqcup_{q=1}^{k^c} \sqcup_{\ddot{\imath}=1}^{s_q^{c,1}} \boldsymbol{\nu}_{q,\ddot{\imath}}^{c,1} & \longrightarrow & \mathbf{j}^{c,1} & = & \sqcup_{q=1}^{k^c} \mathbf{s}_q^{c,1} & \nearrow & \dots \\
 & \dots & & & \dots & \dots & \\
\sqcup_{q=1}^{k^1} \sqcup_{\ddot{\imath}=1}^{s_q^{1,l^1}} \boldsymbol{\nu}_{q,\ddot{\imath}}^{1,l^1} & \longrightarrow & \mathbf{j}^{1,l^1} & = & \sqcup_{q=1}^{k^1} \mathbf{s}_q^{1,l^1} & \longrightarrow & \mathbf{k}^1 \\
 & \dots & & & \dots & & \\
\sqcup_{q=1}^{k^1} \sqcup_{\ddot{\imath}=1}^{s_q^{1,1}} \boldsymbol{\nu}_{q,\ddot{\imath}}^{1,1} & \longrightarrow & \mathbf{j}^{1,1} & = & \sqcup_{q=1}^{k^1} \mathbf{s}_q^{1,1} & \nearrow &
\end{array}$$

Respectively, left action (1.26) of operads $\mathcal{A}_g^c$ on $\circledast_{\mathsf{G}}(t)((\mathcal{P}^c)_{c\in\mathbf{n}}, \mathcal{Q})$ is given on the direct summand corresponding to τ^0_+ by

$$\begin{aligned}
&\odot_{>0}\big((\mathcal{A}_g^c)_{g\in\mathbf{l}^c}^{c\in\mathbf{n}}; \circledast_{\mathsf{G}}(t)((\mathcal{P}^c)_{c\in\mathbf{n}}, \mathcal{Q})\big)(\tau^0_+) \\
&\quad = \Big[\bigotimes^{c\in\mathbf{n}}\bigotimes^{g\in\mathbf{l}^c}\bigotimes^{q\in\mathbf{k}^c}\bigotimes^{\ddot{\imath}\in\mathbf{s}_q^{c,g}} \mathcal{A}_g^c(\boldsymbol{\nu}_{q,\ddot{\imath}}^{c,g})\Big] \otimes \Big(\bigotimes^{c\in\mathbf{n}}\bigotimes^{q\in\mathbf{k}^c} \mathcal{P}^c\big((s_q^{c,g})_{g\in\mathbf{l}^c}\big)\Big) \otimes \mathcal{Q}(k) \\
&\quad \xrightarrow[\sim]{\lambda_{\psi_2}} \Big\{\bigotimes^{c\in\mathbf{n}}\bigotimes^{q\in\mathbf{k}^c}\Big[\Big(\bigotimes^{g\in\mathbf{l}^c}\bigotimes^{\ddot{\imath}\in\mathbf{s}_q^{c,g}} \mathcal{A}_g^c(\boldsymbol{\nu}_{q,\ddot{\imath}}^{c,g})\Big) \otimes \mathcal{P}^c\big((s_q^{c,g})_{g\in\mathbf{l}^c}\big)\Big]\Big\} \otimes \mathcal{Q}(k) \\
&\qquad\qquad \xrightarrow{(\otimes\otimes\lambda)\otimes 1} \Big[\bigotimes^{c\in\mathbf{n}}\bigotimes^{q\in\mathbf{k}^c} \mathcal{P}^c\Big(\Big(\sum_{\ddot{\imath}=1}^{s_q^{c,g}} \nu_{q,\ddot{\imath}}^{c,g}\Big)_{g\in\mathbf{l}^c}\Big)\Big] \otimes \mathcal{Q}(k).
\end{aligned}$$

2. Morphisms with several entries

Here we give support to the observation that morphisms with n entries of algebras over operads form an $n\wedge 1$-operad module. In particular, we find this module for A_∞-algebras.

2.1. Main source of $n \wedge 1$-operad modules. Starting with an arbitrary symmetric **dg**-multicategory C we get a **dg**-operad $\mathcal{E}(X) = \mathcal{E}nd\, X$ for any object X and an $n \wedge 1$-module $\mathcal{H}om = (\mathcal{E}nd\, A_1, \dots, \mathcal{E}nd\, A_n; \mathcal{H}; \mathcal{E}nd\, B)$ for any family $A_1, \dots, A_n, B$ in $\operatorname{Ob}\mathsf{C}$

$$(\mathcal{E}nd\, X)(v) = \mathsf{C}({}^vX; X),$$
$$\mathcal{H}(j^1, \dots, j^n) = \mathcal{H}om(A_1, \dots, A_n; B)(j^1, \dots, j^n) = \mathsf{C}\big(({}^{j^i}A_i)_{i=1}^n; B\big).$$

The right action

$$\begin{aligned}
\rho_{(j_p^i)} : \Big[\bigotimes_{p\in\mathbf{k}} \mathcal{H}\big((j_p^i)_{i\in\mathbf{n}}\big)\Big] \otimes (\mathcal{E}nd\, B)(k) = \Big[\bigotimes_{p\in\mathbf{k}} \mathsf{C}\big(({}^{j_p^i}A_i)_{i=1}^n; B\big)\Big] \otimes \mathsf{C}({}^kB; B) \\
\to \mathsf{C}\big(({}^{\ell^i}A_i)_{i=1}^n; B\big) = \mathcal{H}\big((\ell^i)_{i=1}^n\big),
\end{aligned}$$

where $\ell^i = \sum_{p=1}^k j_p^i$, equals to the multicategory composition μ_ϕ, which corresponds to the map $\phi : \mathbf{l}^1 \sqcup \dots \sqcup \mathbf{l}^n \to \mathbf{k}$, whose restriction to $\mathbf{l}^i$ is isotonic and sends exactly j_p^i elements to $p \in \mathbf{k}$.

The left action

$$\lambda_{(j_p^i)} : \Big[\bigotimes_{i\in\mathbf{n}} \bigotimes_{p=1}^{k^i} (\mathcal{E}nd\, A_i)(j_p^i)\Big] \otimes \mathcal{H}\big((k^i)_{i=1}^n\big) \to \mathcal{H}\Big(\Big(\sum_{p=1}^{k^i} j_p^i\Big)_{i=1}^n\Big),$$

that is,

$$\lambda_{(j_p^i)} : \Big[\bigotimes_{i\in\mathbf{n}} \bigotimes_{p=1}^{k^i} \mathsf{C}({}^{j_p^i}A_i; A_i)\Big] \otimes \mathsf{C}\big(({}^{k^i}A_i)_{i=1}^n; B\big) \to \mathsf{C}\big(({}^{\ell^i}A_i)_{i=1}^n; B\big)$$

with $\ell^i = \sum_{p=1}^{k^i} j_p^i$, equals to the multicategory composition μ_ψ, corresponding to the isotonic map $\psi = \sqcup \sqcup \triangleright : \sqcup_{i=1}^n \sqcup_{p=1}^{k^i} \mathbf{j}_p^i \to \sqcup_{i=1}^n \sqcup_{p=1}^{k^i} \mathbf{1}$, which sends exactly j_p^i elements to the element of the target indexed by (i,p). Notice that $\rho_\varnothing : (\mathcal{E}nd\, Y)(0) = \mathsf{C}(;Y) = \mathcal{H}(0,\dots,0)$ is the identity map.

2.2 Example. In particular, reasonings of Section 2.1 apply to the symmetric **dg**-multicategory $\mathsf{C} = \underline{\mathsf{C}}_\Bbbk$ and for any $(n+1)$-tuple $(X_1,\dots,X_n;Y)$ of complexes give an $n\wedge 1$-operad module

$$(\mathcal{E}nd\, X_1,\dots,\mathcal{E}nd\, X_n; \mathcal{H}om(X_1,\dots,X_n;Y); \mathcal{E}nd\, Y),$$
$$\mathcal{H}(j^1,\dots,j^n) = \mathcal{H}om(X_1,\dots,X_n;Y)(j^1,\dots,j^n) = \underline{\mathsf{C}}_\Bbbk\big(({}^{j^i}X_i)_{i=1}^n; Y\big).$$

The case of $n=0$ gives $\mathcal{H} = Y$. The left action

$$\lambda_{(j_p^i)} : \Big[\bigotimes_{i\in\mathbf{n}} \bigotimes_{p=1}^{k^i} \underline{\mathsf{C}}_\Bbbk({}^{j_p^i}A_i; A_i)\Big] \otimes \underline{\mathsf{C}}_\Bbbk\big(({}^{k^i}A_i)_{i=1}^n; B\big) \to \underline{\mathsf{C}}_\Bbbk\big(({}^{\ell^i}A_i)_{i=1}^n; B\big), \quad (\otimes_i \otimes_p g_p^i) \otimes f \mapsto h$$

with $\ell^i = \sum_{p=1}^{k^i} j_p^i$ is found as

$$h = \big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i}\mathcal{A}_i \xrightarrow{\boxtimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}^i} T^{j_p^i}\mathcal{A}_i \xrightarrow{\boxtimes^{i\in\mathbf{n}}\otimes^{p\in\mathbf{k}^i} g_p^i} \boxtimes^{i\in\mathbf{n}} T^{k^i}\mathcal{A}_i \xrightarrow{f} \mathcal{B}\big]$$

For the scope of this article there is no distinction between $\boxtimes$ and $\otimes$, see Section 0.2. The right action

$$\rho_{(j_p^i)} \;:\; \Big[\bigotimes_{p\in\mathbf{k}} \underline{\mathsf{C}}_\Bbbk\big(({}^{j_p^i}A_i)_{i=1}^n; B\big)\Big] \otimes \underline{\mathsf{C}}_\Bbbk({}^kB; B) \;\to\; \underline{\mathsf{C}}_\Bbbk\big(({}^{\ell^i}A_i)_{i=1}^n; B\big), \quad (\otimes_p f^p) \otimes g \;\mapsto\; h,$$

where $\ell^i = \sum_{p=1}^k j_p^i$, is found as (see [BLM08, Eq. (6.1.1)] for isomorphism $\overline{\varkappa}$)

$$h = \big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i}A_i \xrightarrow{\boxtimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j_p^i}A_i \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{k}} \boxtimes^{i\in\mathbf{n}} T^{j_p^i}A_i \xrightarrow{\otimes^{p\in\mathbf{k}} f^p} \otimes^{p\in\mathbf{k}} \mathcal{B} \xrightarrow{g} \mathcal{B}\big].$$

Given an operad $\mathcal{O}$ and an $n \wedge 1$-operad $\mathcal{O}$-module $\mathcal{F}_n$ for each $n \geqslant 0$ we define a morphism of $\mathcal{O}$-algebras with n arguments $X_1, \dots, X_n \to Y$ as a morphism of ${}_n\mathrm{Op}_1$

$$(\mathcal{O}, \dots, \mathcal{O}; \mathcal{F}_n; \mathcal{O}) \to (\mathcal{E}nd\, X_1, \dots, \mathcal{E}nd\, X_n; \mathit{hom}(X_1, \dots, X_n; Y); \mathcal{E}nd\, Y).$$

2.3 Example. Produce an $n \wedge 1$-operad As-module FAs_n from the symmetric **dg**-multicategory Com with one object $*$ – the symmetric **dg**-operad of associative non–unital commutative algebras. It has $Com(k) = \Bbbk$ for $k > 0$, and $Com(0) = 0$. The compositions are given by multiplication in $\Bbbk$. Hence, $\mathcal{E}nd_{Com} * = As$ and $FAs_n = \mathit{hom}_{Com}({}^n*; *)$ has $FAs_n(j^1, \dots, j^n) = \Bbbk = \Bbbk u_j$ for all non–vanishing $(j^1, \dots, j^n) \in \mathbb{N}^n$, while $FAs_n(0, \dots, 0) = 0$. In particular, $FAs_0 = 0$. The actions for FAs_n are given by multiplication in $\Bbbk$. A morphism of $n \wedge 1$-operad modules

$$(As, \dots, As; FAs_n; As) \to (\mathcal{E}nd\, A_1, \dots, \mathcal{E}nd\, A_n; \mathit{hom}(A_1, \dots, A_n; B); \mathcal{E}nd\, B)$$

amounts to a family of morphisms $f_i : A_i \to B$ of associative differential graded $\Bbbk$-algebras without units, $i \in \mathbf{n}$, such that the following diagrams commute for all $1 \leqslant i < j \leqslant n$:

$$\begin{tikzcd}
A_i \otimes A_j \arrow[r, "c", "\sim"'] \arrow[d, "f_i \otimes f_j"'] & A_j \otimes A_i \arrow[r, "f_j \otimes f_i"] & B \otimes B \arrow[d, "m_B"] \\
B \otimes B \arrow[rr, "m_B"] & & B
\end{tikzcd} \tag{2.1}$$

In fact, morphisms $f_i = f_{(e_i)}$ are images of the unit under the action map

$$\dot{f}_{(e_i)} : \Bbbk = FAs_n(e_i) \to \underline{\mathsf{C}}_{\Bbbk}(A_i; B), \quad 1 \mapsto f_{(e_i)},$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^n$ has 1 on i-th place. The equations hold for all $1 \leqslant i < j \leqslant n$:

$$\begin{tikzcd}
FAs_n(e_i) \otimes FAs_n(e_j) \otimes As(2) \arrow[r, "\mathrm{mult}"] \arrow[d] \arrow[dr, phantom, "="] & FAs_n(e_i + e_j) \arrow[d] \\
\mathcal{H}(e_i) \otimes \mathcal{H}(e_j) \otimes (\mathcal{E}nd\, B)(2) \arrow[r, "\mu_{\mathrm{id}}"] & \mathcal{H}(e_i + e_j)
\end{tikzcd}$$

$$\begin{tikzcd}
FAs_n(e_j) \otimes FAs_n(e_i) \otimes As(2) \arrow[r, "\mathrm{mult}"] \arrow[d] \arrow[dr, phantom, "="] & FAs_n(e_i + e_j) \arrow[d] \\
\mathcal{H}(e_j) \otimes \mathcal{H}(e_i) \otimes (\mathcal{E}nd\, B)(2) \arrow[r, "\mu_{(12)}"] & \mathcal{H}(e_i + e_j)
\end{tikzcd}$$

Here the compositions μ_{id} and $\mu_{(12)}$ in $\underline{\mathsf{C}}_{\Bbbk}$ correspond to the two maps

$$\mathrm{id} : \mathbf{0} \sqcup \dots \sqcup \mathbf{0} \sqcup \mathbf{1} \sqcup \mathbf{0} \sqcup \dots \sqcup \mathbf{0} \sqcup \mathbf{1} \sqcup \mathbf{0} \sqcup \dots \sqcup \mathbf{0} = \mathbf{2} \to \mathbf{2},$$
$$(12) : \mathbf{0} \sqcup \dots \sqcup \mathbf{0} \sqcup \mathbf{1} \sqcup \mathbf{0} \sqcup \dots \sqcup \mathbf{0} \sqcup \mathbf{1} \sqcup \mathbf{0} \sqcup \dots \sqcup \mathbf{0} = \mathbf{2} \to \mathbf{2}.$$

The equations are more explicit in the form

$$\begin{array}{ccc}
\Bbbk\otimes\Bbbk\otimes\Bbbk & \xrightarrow{\qquad\qquad\text{mult}\qquad\qquad} & \Bbbk \\
{\scriptstyle \dot f_{(e_i)}\otimes\dot f_{(e_j)}\otimes\dot m_B}\big\downarrow & = & \big\downarrow{\scriptstyle \dot f_{(e_i+e_j)}} \\
\underline{\mathsf{C}}_{\Bbbk}(A_i;B)\otimes\underline{\mathsf{C}}_{\Bbbk}(A_j;B)\otimes\underline{\mathsf{C}}_{\Bbbk}(B,B;B) & \xrightarrow{\mu_{\mathrm{id}}} & \underline{\mathsf{C}}_{\Bbbk}(A_i,A_j;B)
\end{array}$$

$$\begin{array}{ccc}
\Bbbk\otimes\Bbbk\otimes\Bbbk & \xrightarrow{\qquad\qquad\text{mult}\qquad\qquad} & \Bbbk \\
{\scriptstyle \dot f_{(e_j)}\otimes\dot f_{(e_i)}\otimes\dot m_B}\big\downarrow & = & \big\downarrow{\scriptstyle \dot f_{(e_i+e_j)}} \\
\underline{\mathsf{C}}_{\Bbbk}(A_j;B)\otimes\underline{\mathsf{C}}_{\Bbbk}(A_i;B)\otimes\underline{\mathsf{C}}_{\Bbbk}(B,B;B) & \xrightarrow{\mu_{(12)}} & \underline{\mathsf{C}}_{\Bbbk}(A_i,A_j;B)
\end{array}$$

The same equations can be written as

$$(f_{(e_i)}\otimes f_{(e_j)})m_B = f_{(e_i+e_j)} = c(f_{(e_j)}\otimes f_{(e_i)})m_B : A_i\otimes A_j\to B,$$

which coincides with condition (2.1).

Notice that if A_i, $i\in\mathbf{n}$, B are unital **dg**-algebras, a collection of unital morphisms $f_i : A_i\to B$ that satisfy equation (2.1) is the same as a single unital morphism $f : A_1\otimes\cdots\otimes A_n\to B$. In fact, such f determines $f_i(x) = f(1\otimes\cdots\otimes 1\otimes x\otimes 1\otimes\cdots\otimes 1)$ and can be recovered from the whole collection of f_i's.

So defined multimorphisms turn the class of associative non–unital **dg**-algebras into a multicategory As. In fact, any totally ordered finite set I can be used instead of $\mathbf{n}$. Corresponding to an isotonic map $\phi : I\to J$, composition of multimorphisms $(f^j : (A_i)_{i\in\phi^{-1}j}\to B_j) = (f^j_i : A_i\to B_j)_{i\in\phi^{-1}j}$, $j\in J$, $(g : (B_j)_{j\in J}\to C) = (g_j : B_j\to C)_{j\in J}$ is given by

$$[(f^j : (A_i)_{i\in\phi^{-1}j}\to B_j)_{j\in J}\cdot(g : (B_j)_{j\in J}\to C)]_k = f^{\phi(k)}_k\cdot g_{\phi(k)} : A_k\to C.$$

One can check condition (2.1) for this collection of morphisms.

2.4 Definition. An $n\wedge 1$-operad module homomorphism

$$(f_1,\dots,f_n;h;g) : (\mathcal{A}_1,\dots,\mathcal{A}_n;\mathcal{P};\mathcal{B})\to(\mathcal{C}_1,\dots,\mathcal{C}_n;\mathcal{Q};\mathcal{D}),$$

of degree $r\in\mathbb{Z}$ is a family of **dg**-operad homomorphisms $g : \mathcal{B}\to\mathcal{D}$, $f_i : \mathcal{A}_i\to\mathcal{C}_i$, $0\leqslant i\leqslant n$, of degree r and a collection of homogeneous $\Bbbk$-linear maps $h(j) : \mathcal{P}(j)\to\mathcal{Q}(j)$, $j\in\mathbb{N}^n$, of degree $r(1-|j|)$ such that

- for all $l\in\mathbb{N}$, $(k_q\in\mathbb{N}^n\mid 1\leqslant q\leqslant l)$, the following square commutes up to the sign

$$\begin{array}{ccc}
\Big(\bigotimes_{q=1}^{l}\mathcal{P}(k_q)\Big)\otimes\mathcal{B}(l) & \xrightarrow{\rho} & \mathcal{P}\Big(\sum_{q=1}^{l}k_q\Big) \\
{\scriptstyle (\otimes_{q=1}^{l}h(k_q))\otimes g(l)}\big\downarrow & (-1)^{c_\rho} & \big\downarrow{\scriptstyle h(\sum_{q=1}^{l}k_q)} \\
\Big(\bigotimes_{q=1}^{l}\mathcal{Q}(k_q)\Big)\otimes\mathcal{D}(l) & \xrightarrow{\rho} & \mathcal{Q}\Big(\sum_{q=1}^{l}k_q\Big)
\end{array}\tag{2.2}$$

$$c_\rho = r\sum_{q=1}^{l}(q-1)(1-|k_q|) + r\sum_{1\leqslant b<a\leqslant l}^{1\leqslant c<d\leqslant n} k_a^c k_b^d$$
$$+\frac{r(r-1)}{2}\Big\{(1-l)\sum_{q=1}^{l}(1-|k_q|) + \sum_{1\leqslant q<s\leqslant l}(1-|k_q|)(1-|k_s|)\Big\}; \quad (2.3)$$

- for all $k\in\mathbb{N}^n$, $(j_p^i\in\mathbb{N}\mid 1\leqslant i\leqslant n,\ 0\leqslant p\leqslant k^i)$, the following square commutes up to the sign

$$\begin{tikzcd}
\Big[\bigotimes_{i=1}^{n}\bigotimes_{p=1}^{k^i}\mathcal{A}_i(j_p^i)\Big]\otimes\mathcal{P}\big((k^i)_{i=1}^n\big) \arrow[r, "\lambda"] \arrow[d, "{[\otimes_{i=1}^n\otimes_{p=1}^{k^i}f_i(j_p^i)]\otimes h(k)}"'] \arrow[dr, phantom, "(-1)^{c_\lambda}"] & \mathcal{P}\Big(\Big(\sum_{p=1}^{k^i}j_p^i\Big)_{i=1}^n\Big) \arrow[d, "{h((\sum_{p=1}^{k^i}j_p^i)_{i=1}^n)}"] \\
\Big[\bigotimes_{i=1}^{n}\bigotimes_{p=1}^{k^i}\mathcal{C}_i(j_p^i)\Big]\otimes\mathcal{Q}\big((k^i)_{i=1}^n\big) \arrow[r, "\lambda"] & \mathcal{Q}\Big(\Big(\sum_{p=1}^{k^i}j_p^i\Big)_{i=1}^n\Big)
\end{tikzcd} \quad (2.4)$$

$$c_\lambda = r\sum_{i=1}^{n}\sum_{p=1}^{k^i}(1-j_p^i)\Big(p-1+\sum_{q=1}^{i-1}k^q\Big) + \frac{r(r-1)}{2}\Big\{(1-|k|)\sum_{i=1}^{n}\sum_{p=1}^{k^i}(1-j_p^i)$$
$$+\sum_{1\leqslant i<l\leqslant n}\Big[\sum_{p=1}^{k^i}(1-j_p^i)\Big]\Big[\sum_{q=1}^{k^l}(1-j_q^l)\Big] + \sum_{i=1}^{n}\sum_{1\leqslant p<q\leqslant k^i}(1-j_p^i)(1-j_q^i)\Big\};$$

- for all $j\in\mathbb{N}^n$
$$d\cdot h(j) = (-1)^{r(1-|j|)}h(j)\cdot d : \mathcal{P}(j)\to\mathcal{Q}(j).$$

The second (shuffle) part of c_ρ, c_λ proportional to $r(r-1)/2$ makes sure that the composition of morphisms of degrees r and r' be a morphism of degree $r+r'$. The first part coincides with $rc(\tilde\tau_\rho)$, $rc(\tilde\tau_\lambda)$.

The last condition using λ can be replaced with n conditions using λ^i, $1\leqslant i\leqslant n$:

$$\begin{tikzcd}
\Big[\bigotimes_{p=1}^{k^i}\mathcal{A}_i(j_p)\Big]\otimes\mathcal{P}(k) \arrow[r, "\lambda^i"] \arrow[d, "{[\otimes_{p=1}^{k^i}f_i(j_p)]\otimes h(k)}"'] \arrow[dr, phantom, "(-1)^{c_{\lambda^i}}"] & \mathcal{P}\Big(k, k^i\mapsto\sum_{p=1}^{k^i}j_p\Big) \arrow[d, "{h(k,k^i\mapsto\sum_{p=1}^{k^i}j_p)}"] \\
\Big[\bigotimes_{p=1}^{k^i}\mathcal{C}_i(j_p)\Big]\otimes\mathcal{Q}(k) \arrow[r, "\lambda^i"] & \mathcal{Q}\Big(k, k^i\mapsto\sum_{p=1}^{k^i}j_p\Big)
\end{tikzcd}$$

$$c_{\lambda^i} = r\sum_{p=1}^{k^i}(1-j_p)\Big(p-1+\sum_{q=1}^{i-1}k^q\Big) + \frac{r(r-1)}{2}\Big\{(1-|k|)\sum_{p=1}^{k^i}(1-j_p) + \sum_{1\leqslant p<q\leqslant k^i}(1-j_p)(1-j_q)\Big\}. \quad (2.5)$$

2.5 Example. For all complexes $A_1, \dots, A_n$, B the collection

$$\Sigma = ({}^n hom(\sigma;\sigma^{-1}); hom({}^n\sigma;\sigma^{-1}); hom(\sigma;\sigma^{-1})) : \\ \mathcal{H}om(sA_1,\dots,sA_n;sB) \to \mathcal{H}om(A_1,\dots,A_n;B)$$

is an $n \wedge 1$-operad morphism of degree 1. In fact, equations for Σ involving λ and ρ are particular cases of Corollary 1.39.

2.6. A_∞**-morphisms with several entries.** Produce an $n \wedge 1$-operad *As1*-module $FAs1_n$ from the symmetric **dg**-multicategory *Com1* with one object $*$ – the symmetric **dg**-operad of associative unital commutative algebras. It has $Com1(p) = \Bbbk$ for $p \geqslant 0$. The compositions are given by multiplication in $\Bbbk$. Hence, $\mathcal{E}nd_{Com1} * = As1$ and $FAs1_n = hom_{Com1}({}^n*;*)$ has $FAs1_n(j^1,\dots,j^n) = \Bbbk$ for all $(j^1,\dots,j^n) \in \mathbb{N}^n$. In particular, $FAs1_0 = \Bbbk$. The actions for $FAs1_n$ are given by multiplication in $\Bbbk$. A morphism of $n \wedge 1$-operad modules

$$(As1,\dots,As1; FAs1_n; As1) \to (\mathcal{E}nd\, A_1,\dots,\mathcal{E}nd\, A_n; hom(A_1,\dots,A_n;B); \mathcal{E}nd\, B)$$

amounts to a family of unital morphisms $f_i : A_i \to B$ of associative unital differential graded $\Bbbk$-algebras, $i \in \mathbf{n}$, such that diagrams (2.1) commute for all $1 \leqslant i < j \leqslant n$. These data are in bijection with unital homomorphisms $f : A_1 \otimes \dots \otimes A_n \to B$, where $A_1, \dots, A_n$, B are unital associative **dg**-algebras.

In fact, each complex $A_1, \dots, A_n$, B acquires a unital associative **dg**-algebra structure through morphisms $As1 \to \mathcal{E}nd\, A_i$, $As1 \to \mathcal{E}nd\, B$. Particular cases of actions

$$\lambda_{e_i} : \mathcal{A}_i(0) \otimes \mathcal{P}(e_i) \to \mathcal{P}(0),$$
$$\rho_\varnothing : \mathcal{B}(0) = \Bbbk \otimes \mathcal{B}(0) \to \mathcal{P}(0),$$

for the module $(As1,\dots,As1; FAs1_n; As1)$ take unity to unity:

$$\lambda_{e_i} : As1(0) \otimes FAs1_n(e_i) \ni 1 \otimes 1 \mapsto 1 \in FAs1_n(0), \\ \rho_\varnothing : As1(0) \ni 1 \mapsto 1 \in FAs1_n(0).$$

Commutative diagram [Lyu11, (2.2)] with $\mathcal{H}om(A_1,\dots,A_n;B)(0)$ in place of $\mathcal{H}om(A;B)(0)$ shows that $1 \in FAs1_n(0)$ is represented by 1_B. Since the representation agrees with λ_{e_i} the equation $1_{A_i}.f_i = 1_B$ holds, thus, f_i is unital.

2.7 Proposition. *There is the $n \wedge 1$-operad A_∞-module $F_n = \boxdot_{\geqslant 0}({}^nA_\infty; \Bbbk\{f_j \mid j \in \mathbb{N}^n - 0\}; A_\infty)$ freely generated as a graded module by elements $f_{j^1,\dots,j^n} \in F_n(j^1,\dots,j^n)$, $(j^1,\dots,j^n) \in \mathbb{N}^n - 0$, of degree 0. The differential for it is given by*

$$f_\ell\partial = \sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>1} \lambda^q_{({}^r1,x,{}^t1)}({}^r1, b_x, {}^t1; f_{\ell-(x-1)e_q}) - \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n-0 \\ j_1+\dots+j_k=\ell}}^{k>1} \rho_{(j^i_p)}((f_{j_p})^k_{p=1}; b_k). \tag{2.6}$$

The first arguments of λ are all $1 \in A_\infty(1)$ except b_x on the only place $p = r+1$. F_n-maps are A_∞-algebra morphisms $A_1,\dots,A_n \to B$ (for algebras written with operations b_n).

Proof. Notice that $F_0 = A_\infty(0) = 0$ by Lemma 1.34.

The following lemma is verified straightforwardly.

2.8 Lemma. *For* **dg**-*operads* $\mathcal{A}_1, \dots, \mathcal{A}_n$ *there is a* **dg**-*category* $\mathcal{A}_1\text{-}\cdots\text{-}\mathcal{A}_n\text{-}\mathrm{mod}$, *whose objects are left* n-*operad* $\mathcal{A}_1\text{-}\cdots\text{-}\mathcal{A}_n$-*modules and degree* t *morphisms* $f : \mathcal{P} \to \mathcal{Q}$ *are collections of* $\Bbbk$-*linear maps* $f(k^1, \dots, k^n) : \mathcal{P}(k^1, \dots, k^n) \to \mathcal{Q}(k^1, \dots, k^n)$ *of degree* t *such that*

$$\begin{array}{ccc}
\Big[\bigotimes_{i\in\mathbf{n}}\bigotimes_{p=1}^{k^i}\mathcal{A}_i(j^i_p)\Big]\otimes\mathcal{P}\big((k^i)_{i=1}^n\big) & \xrightarrow{\lambda_{(j^i_p)}} & \mathcal{P}\Big(\Big(\sum_{p=1}^{k^i} j^i_p\Big)_{i=1}^n\Big) \\
{\scriptstyle [\otimes\otimes 1]\otimes f}\Big\downarrow & = & \Big\downarrow{\scriptstyle f} \\
\Big[\bigotimes_{i\in\mathbf{n}}\bigotimes_{p=1}^{k^i}\mathcal{A}_i(j^i_p)\Big]\otimes\mathcal{Q}\big((k^i)_{i=1}^n\big) & \xrightarrow{\lambda_{(j^i_p)}} & \mathcal{Q}\Big(\Big(\sum_{p=1}^{k^i} j^i_p\Big)_{i=1}^n\Big)
\end{array}$$

The differential is $f \mapsto [f, \partial] = f\partial - (-1)^f \partial f$.

A *connection* on a graded $n \wedge 1$-operad module $\mathcal{P}$ over **dg**-operads $\mathcal{A}_1, \dots, \mathcal{A}_n$, $\mathcal{B}$ is a collection of $\Bbbk$-linear maps $\partial : \mathcal{P}(j) \to \mathcal{P}(j)$ of degree 1, $j \in \mathbb{N}^n$, which can be viewed as functors $\mathbb{Z} \to \Bbbk\text{-mod}$, $p \mapsto \mathcal{P}(j)^p$, where the category $\mathbb{Z}$ comes from the ordered set $\mathbb{Z}$. All action maps λ^i, ρ from (1.18) are required to be natural (chain) transformations with respect to the sum of maps $1^{\otimes a} \otimes \partial \otimes 1^{\otimes b}$ in the source, where ∂ denotes the connection on the module or the differential in an operad. Equivalently, action maps λ, ρ are chain transformations, or, equivalently, action maps α from (1.17) are chain transformations. A connection on a freely generated module $\mathcal{P}$ is unambiguously fixed by its value on generators.

The square ∂^2 of a connection ∂ is also a connection (of degree 2). It makes all actions into chain transformations with respect to the sum of maps $1^{\otimes a} \otimes \partial^2 \otimes 1^{\otimes b}$ in the source (where ∂^2 vanishes if applied to an operad). In particular, $\partial^2 : \mathcal{P} \to \mathcal{P}$ is a morphism of graded left n-operad $\mathcal{A}_1\text{-}\cdots\text{-}\mathcal{A}_n$-modules of degree 2 as defined in Lemma 2.8. If ∂^2 vanishes, $(\mathcal{P}, \partial)$ becomes an $n \wedge 1$-operad **dg**-module.

2.9 Lemma. F_n *is an* $n \wedge 1$-*operad* **dg**-*module.*

Proof. Recall that the differential in the operad A_∞ is given by

$$b_n.\partial = -\sum_{a+p+c=n}^{p>1,\ a+c>0} \mu({}^a 1, b_p, {}^c 1; b_{a+1+c}).$$

Let us prove that $\partial^2 = 0$ for connection ∂ given by (2.6). Let us verify this on generators:

$$f_\ell \partial^2 = \sum_{q=1}^{n} \sum_{k+y+m=\ell^q}^{y>1} \lambda^q({}^k1, b_y, {}^m1; f_{\ell-(y-1)e_q}.\partial) + \sum_{q=1}^{n} \sum_{u+c+v=\ell^q}^{c>1} \lambda^q({}^u1, b_c.\partial, {}^v1; f_{\ell-(c-1)e_q})$$

$$- \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} \rho((f_{j_p})_{p=1}^k; b_k.\partial) + \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} \sum_{p=1}^{k} \rho(f_{j_1}, \dots, f_{j_{p-1}}, f_{j_p}.\partial, f_{j_{p+1}}, \dots, f_{j_k}; b_k)$$

$$= \sum_{q=1}^{n} \sum_{k+y+m=\ell^q}^{y>1} \sum_{p=1}^{n} \sum_{u+h+v=\ell^q-y+1}^{y>1} \lambda^q({}^k1, b_y, {}^m1; \lambda^p({}^u1, b_h, {}^v1; f_{\ell-(y-1)e_q-(h-1)e_p})) \tag{2.7}$$

$$- \sum_{q=1}^{n} \sum_{c>1} \sum_{r+c+t=\ell^q} \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell-(c-1)e_q}}^{k>1} \lambda^q({}^r1, b_c, {}^t1; \rho((f_{j_p})_p; b_k)) \tag{2.8}$$

$$- \sum_{q=1}^{n} \sum_{u+c+v=\ell^q}^{c>1} \sum_{x+y+z=c}^{y>1} \lambda^q({}^{u+x}1, b_y, {}^{z+v}1; \lambda^q({}^u1, b_{x+1+z}, {}^v1; f_{\ell-(c-1)e_q})) \tag{2.9}$$

$$- \sum_{\substack{j_1,\dots,j_s \in \mathbb{N}^n - 0 \\ j_1+\dots+j_s=\ell}}^{s>1} \sum_{x+m+z=s}^{m>1} \rho((f_{j_p})_{p=1}^s; \mu({}^x1, b_m, {}^z1; b_{x+1+z})) \tag{2.10}$$

$$+ \sum_{q=1}^{n} \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} \sum_{p=1}^{k} \sum_{x+c+z=j_p^q}^{c>1} \rho(f_{j_1}, \dots, f_{j_{p-1}}, \lambda^q({}^x1, b_c, {}^z1; f_{j_p-(c-1)e_q}), f_{j_{p+1}}, \dots, f_{j_k}; b_k) \tag{2.11}$$

$$- \sum_{\substack{y_1,\dots,y_k \in \mathbb{N}^n - 0 \\ y_1+\dots+y_k=\ell}}^{k>1} \sum_{p=1}^{k} \sum_{\substack{t_1,\dots,t_m \in \mathbb{N}^n - 0 \\ t_1+\dots+t_m=y_p}}^{m>1} \rho(f_{y_1}, \dots, f_{y_{p-1}}, \rho(f_{t_1}, \dots, f_{t_m}; b_m), f_{y_{p+1}}, \dots, f_{y_k}; b_k). \tag{2.12}$$

Summands of (2.7) pairwise cancel each other if $p \neq q$. Also summands of (2.7) with $p = q$ pairwise cancel each other if output of b_y does not become an input of b_h. The remainder of (2.7) cancels with sum (2.9). Sums (2.8) and (2.11) cancel each other. Identifying in sums (2.10) and (2.12) the index s with $k + m - 1$ and the sequence $(j_1, \dots, j_s)$ with the sequence $(y_1, \dots, y_{p-1}, t_1, \dots, t_m, y_{p+1}, \dots, y_k)$ we see that they cancel. Therefore, ∂^2 vanishes. ∎

The image of $f_\ell \partial$ in $hom((sA_i)_i; sB)$ is

$$\sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>1} \big[\boxtimes^{i \in \mathbf{n}} T^{\ell^i} sA_i \xrightarrow{1^{\boxtimes(q-1)} \boxtimes (1^{\otimes r} \otimes b_x \otimes 1^{\otimes t}) \boxtimes 1^{\boxtimes(n-q)}}$$

$$T^{\ell^1} sA_1 \boxtimes \cdots \boxtimes T^{\ell^{q-1}} sA_{q-1} \boxtimes T^{r+1+t} sA_q \boxtimes T^{\ell^{q+1}} sA_{q+1} \boxtimes \cdots \boxtimes T^{\ell^n} sA_n \xrightarrow{f_{\ell-(x-1)e_q}} sB\big]$$

$$- \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} \big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{\boxtimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j_p^i} sA_i \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{k}} \boxtimes^{i\in\mathbf{n}} T^{j_p^i} sA_i$$

$$\xrightarrow{\otimes^{p\in\mathbf{k}} f_{j_p}} \otimes^{p\in\mathbf{k}} sB \xrightarrow{b_k} sB\big].$$

Isomorphisms λ^{γ_i} and $\overline{\varkappa}$ are the obvious ones, see [BLM08] for details.

An F_n-algebra map is specified by A_∞-algebras $A_1, \dots, A_n$, B, and a collection of $\Bbbk$-linear degree 0 maps $f_j : \boxtimes^{i\in\mathbf{n}} T^{j^i} sA_i \to sB$ assigned to generators $(f_j)_{j\in\mathbb{N}^n-0}$. It suffices to satisfy on generators the only requirement that $F_n \to hom((A_i[1])_{i=1}^n; B[1])$ be a chain map. The latter means that the equation holds for all $\ell \in \mathbb{N}^n - 0$:

$$f_\ell b_1 - \Bigg[\sum_{q=1}^{n} \sum_{r+1+t=\ell^q} 1^{\boxtimes(q-1)} \boxtimes (1^{\otimes r} \otimes b_1 \otimes 1^{\otimes t}) \boxtimes 1^{\boxtimes(n-q)}\Bigg] f_\ell = f_\ell \partial.$$

Explicitly this equation says

$$\sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>0} \big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{1^{\boxtimes(q-1)}\boxtimes(1^{\otimes r}\otimes b_x \otimes 1^{\otimes t})\boxtimes 1^{\boxtimes(n-q)}}$$

$$T^{\ell^1} sA_1 \boxtimes \cdots \boxtimes T^{\ell^{q-1}} sA_{q-1} \boxtimes T^{r+1+t} sA_q \boxtimes T^{\ell^{q+1}} sA_{q+1} \boxtimes \cdots \boxtimes T^{\ell^n} sA_n \xrightarrow{f_{\ell-(x-1)e_q}} sB\big]$$

$$= \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>0} \big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i} sA_i \xrightarrow{\boxtimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j_p^i} sA_i$$

$$\xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{k}} \boxtimes^{i\in\mathbf{n}} T^{j_p^i} sA_i \xrightarrow{\otimes^{p\in\mathbf{k}} f_{j_p}} \otimes^{p\in\mathbf{k}} sB \xrightarrow{b_k} sB\big]. \quad (2.13)$$

Collections $(f_j)_{j\in\mathbb{N}^n-0}$ are in bijection with augmented coalgebra morphisms $f : \boxtimes^{i\in\mathbf{n}} TsA_i \to TsB$. Coherence with augmentation means that

$$\big(\Bbbk = \boxtimes^{i\in\mathbf{n}} T^0 sA_i \xrightarrow{f|} TsB\big) = \big(\Bbbk = T^0 sB \hookrightarrow TsB\big).$$

Tensor quivers TsB of A_∞-algebras B are **dg**-coalgebras, whose differential $b : TsB \to TsB$ has the components $b_k : T^k sB \to sB$. Equation (2.13) can be rewritten as

$$\big(\boxtimes^{i\in\mathbf{n}} TsA_i \xrightarrow{f} TsB \xrightarrow{b} sB\big) = \big(\boxtimes^{i\in\mathbf{n}} TsA_i \xrightarrow{\sum_{i=1}^n 1^{\boxtimes(i-1)}\boxtimes b \boxtimes 1^{\boxtimes(n-i)}} \boxtimes^{i\in\mathbf{n}} TsA_i \xrightarrow{f} sB\big).$$

In other terms, f is an augmented **dg**-coalgebra morphism. These are A_∞-morphisms $A_1, \dots, A_n \to B$ by definition, see [BLM08]. □

2.10 Proposition. *There is an $n \wedge 1$-operad module $(\mathrm{A}_\infty, \mathrm{F}_n)$ freely generated as graded module by elements $\mathsf{f}_{j^1,\dots,j^n} \in \mathrm{F}_n(j^1,\dots,j^n)$, $(j^1,\dots,j^n) \in \mathbb{N}^n - 0$, of degree $1 - j^1 - \dots - j^n = 1 - |j|$. The differential for it is given by*

$$\mathsf{f}_\ell \partial = \sum_{q=1}^{n} \sum_{r+x+t=\ell^q}^{x>1} (-1)^{(1-x)(\ell^1+\dots+\ell^{q-1}+r)+1-|\ell|} \lambda^q_{({}^r1,x,{}^t1)}({}^r1, m_x, {}^t1; \mathsf{f}_{\ell-(x-1)e_q})$$
$$+ \sum_{\substack{j_1,\dots,j_k \in \mathbb{N}^n - 0 \\ j_1+\dots+j_k=\ell}}^{k>1} (-1)^{k+\sum_{1\leqslant b<a\leqslant k}^{1\leqslant c<d\leqslant n} j^c_a j^d_b + \sum_{p=1}^k (p-1)(|j_p|-1)} \rho_{(j^i_p)}((\mathsf{f}_{j_p})^k_{p=1}; m_k). \quad (2.14)$$

There is an invertible morphism of degree 1 between these $n \wedge 1$-operad modules

$$(\Sigma, \Sigma) : (A_\infty, F_n) \to (\mathrm{A}_\infty, \mathrm{F}_n), \qquad b_i \mapsto m_i, \quad f_j \mapsto \mathsf{f}_j. \quad (2.15)$$

F_n-maps are A_∞-algebra morphisms $A_1, \dots, A_n \to B$ (for algebras written with operations m_n). The two notions of A_∞-morphisms agree in the sense that the square of $n \wedge 1$-operad module maps

$$\begin{tikzcd}
({}^nA_\infty; F_n; A_\infty) \arrow[r] \arrow[d, "({}^n\Sigma;\Sigma;\Sigma)"'] & ((\mathcal{E}nd\, A_i[1])^n_{i=1}; hom((A_i[1])^n_{i=1}; B[1]); \mathcal{E}nd\, B[1]) \arrow[d, "({}^n hom(\sigma;\sigma^{-1}); hom({}^n\sigma;\sigma^{-1}); hom(\sigma;\sigma^{-1}))"] \\
({}^n\mathrm{A}_\infty; \mathrm{F}_n; \mathrm{A}_\infty) \arrow[r] & ((\mathcal{E}nd\, A_i)^n_{i=1}; hom((A_i)^n_{i=1}; B); \mathcal{E}nd\, B)
\end{tikzcd} \quad (2.16)$$

commutes.

Proof. The existence of F_n implies the existence of F_n as the following lemma shows.

2.11 Lemma. *Let $(\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B})$ be a* **dg**-*$n \wedge 1$-operad module, $(\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{Q}; \mathcal{D})$ be a graded $n \wedge 1$-operad module and*

$$(f_1, \dots, f_n; h; g) : (\mathcal{A}_1, \dots, \mathcal{A}_n; \mathcal{P}; \mathcal{B}) \to (\mathcal{C}_1, \dots, \mathcal{C}_n; \mathcal{Q}; \mathcal{D}),$$

be an invertible graded $n \wedge 1$-operad module homomorphism of degree r (equations (2.2) *and* (2.4) *hold). Then $\mathcal{C}_1$, ..., $\mathcal{C}_n$, $\mathcal{D}$ are* **dg**-*operads (see Remark 0.15) and $\mathcal{P}$ has a unique differential d which turns it into a* **dg**-*$n \wedge 1$-operad module and $(f_1, \dots, f_n; h; g)$ into a* **dg**-*$n \wedge 1$-operad module isomorphism of degree r.*

Proof. The differential is given by a unique expression

$$d = \big(\mathcal{Q}(j) \xrightarrow{h(j)^{-1}} \mathcal{P}(j) \xrightarrow{(-1)^{r(1-|j|)}d} \mathcal{P}(j) \xrightarrow{h(j)} \mathcal{Q}(j)\big).$$

Clearly, $\deg d = 1$ and $d^2 = 0$. Verification that ρ and λ for $\mathcal{Q}$ are chain maps is straightforward. □

Let us compute the value of the differential on generators f_ℓ:

$$
\begin{aligned}
\mathsf{f}_\ell\partial &= (f_\ell.\Sigma(\ell))\partial = (-1)^{1-|\ell|}(f_\ell.\partial)\Sigma(\ell) \\
&= (-1)^{1-|\ell|}\sum_{q=1}^{n}\sum_{r+x+t=\ell^q}^{x>1}\lambda^q_{({}^r1,x,{}^t1)}({}^r1,b_x,{}^t1;f_{\ell-(x-1)e_q}).\Sigma(\ell) \\
&\qquad + (-1)^{|\ell|}\sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>1}\rho_{(j^i_p)}((f_{j_p})^k_{p=1};b_k).\Sigma(\ell) \\
&= \sum_{q=1}^{n}\sum_{r+x+t=\ell^q}^{x>1}(-1)^{c(\tilde\tau_{\lambda^q})+1-|\ell|}\lambda^q_{({}^r1,x,{}^t1)}({}^r1,m_x,{}^t1;\mathsf{f}_{\ell-(x-1)e_q}) \\
&\qquad + \sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>1}(-1)^{k+c(\tilde\tau_\rho)}\rho_{(j^i_p)}((\mathsf{f}_{j_p})^k_{p=1};m_k),
\end{aligned}
$$

which coincides with (2.14), if one plugs in expressions $c(\tilde\tau_{\lambda^q}) = c_{\lambda^q}$ from (2.5) and $c(\tilde\tau_\rho) = c_\rho$ from (2.3) for $r = 1$.

The image of $\mathsf{f}_\ell\partial$ in $\mathcal{hom}((A_i)_i;B)$ is

$$
\begin{aligned}
&\sum_{q=1}^{n}\sum_{r+x+t=\ell^q}^{x>1}(-1)^{(1-x)(\ell^1+\dots+\ell^{q-1}+r)+1-|\ell|}\bigl[\boxtimes^{i\in\mathbf{n}}T^{\ell^i}A_i \xrightarrow{1^{\boxtimes(q-1)}\boxtimes(1^{\otimes r}\otimes m_x\otimes 1^{\otimes t})\boxtimes 1^{\boxtimes(n-q)}} \\
&\qquad T^{\ell^1}A_1\boxtimes\dots\boxtimes T^{\ell^{q-1}}A_{q-1}\boxtimes T^{r+1+t}A_q\boxtimes T^{\ell^{q+1}}A_{q+1}\boxtimes\dots\boxtimes T^{\ell^n}A_n \xrightarrow{\mathsf{f}_{\ell-(x-1)e_q}} B\bigr] \\
&+\sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>1}(-1)^{k+\sum^{1\leqslant c<d\leqslant n}_{1\leqslant b<a\leqslant k}j^c_aj^d_b+\sum_{p=1}^k(p-1)(|j_p|-1)}\bigl[\boxtimes^{i\in\mathbf{n}}T^{\ell^i}A_i \xrightarrow{\boxtimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \boxtimes^{i\in\mathbf{n}}\otimes^{p\in\mathbf{k}}T^{j^i_p}A_i \\
&\qquad\qquad \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{k}}\boxtimes^{i\in\mathbf{n}}T^{j^i_p}A_i \xrightarrow{\otimes^{p\in\mathbf{k}}\mathsf{f}_{j_p}} \otimes^{p\in\mathbf{k}}B \xrightarrow{m_k} B\bigr].
\end{aligned}
$$

F_n-algebra maps consist of A_∞-algebras $A_1, \dots, A_n$, B, and a collection $(\mathsf{f}_j)_{j\in\mathbb{N}^n-0}$ that satisfies the following equation for all $\ell\in\mathbb{N}^n-0$:

$$
\mathsf{f}_\ell m_1 + (-1)^{|\ell|}\Biggl[\sum_{q=1}^{n}\sum_{r+1+t=\ell^q}1^{\boxtimes(q-1)}\boxtimes(1^{\otimes r}\otimes m_1\otimes 1^{\otimes t})\boxtimes 1^{\boxtimes(n-q)}\Biggr]\mathsf{f}_\ell = \mathsf{f}_\ell\partial.
$$

In expanded form the equation says:

$$
\begin{aligned}
&\sum_{q=1}^{n}\sum_{r+x+t=\ell^q}^{x>0}(-1)^{(1-x)(\ell^1+\dots+\ell^{q-1}+r)-|\ell|}\bigl[\boxtimes^{i\in\mathbf{n}}T^{\ell^i}A_i \xrightarrow{1^{\boxtimes(q-1)}\boxtimes(1^{\otimes r}\otimes m_x\otimes 1^{\otimes t})\boxtimes 1^{\boxtimes(n-q)}} \\
&\qquad T^{\ell^1}A_1\boxtimes\dots\boxtimes T^{\ell^{q-1}}A_{q-1}\boxtimes T^{r+1+t}A_q\boxtimes T^{\ell^{q+1}}A_{q+1}\boxtimes\dots\boxtimes T^{\ell^n}A_n \xrightarrow{\mathsf{f}_{\ell-(x-1)e_q}} B\bigr]
\end{aligned}
$$

$$= \sum_{\substack{j_1,\dots,j_k\in\mathbb{N}^n-0\\ j_1+\dots+j_k=\ell}}^{k>0} (-1)^{k+\sum_{1\leqslant b<a\leqslant k}^{1\leqslant c<d\leqslant n} j_a^c j_b^d+\sum_{p=1}^k (p-1)(|j_p|-1)} \Big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i} A_i \xrightarrow{\boxtimes^{i\in\mathbf{n}}\lambda^{\gamma_i}} \boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{k}} T^{j_p^i} A_i$$

$$\xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{k}} \boxtimes^{i\in\mathbf{n}} T^{j_p^i} A_i \xrightarrow{\otimes^{p\in\mathbf{k}}\mathsf{f}_{j_p}} \otimes^{p\in\mathbf{k}} B \xrightarrow{m_k} B\Big]. \quad (2.17)$$

This is actually the definition of an A_∞-algebra morphism $A_1,\dots,A_n \to B$ for algebras written with operations m_n, adopted in the current article.

Relationship between f_j and f_j in $hom((A_i)_{i=1}^n; B)(j)$,

$$\begin{tikzcd} T^{j^1}A_1 \boxtimes \dots \boxtimes T^{j^n}A_n \arrow[r, "\mathsf{f}_j"] \arrow[d, "\sigma^{\otimes j^1}\boxtimes\dots\boxtimes\sigma^{\otimes j^n}"'] & B \arrow[d, "\sigma"] \\ T^{j^1}sA_1 \boxtimes \dots \boxtimes T^{j^n}sA_n \arrow[r, "f_j"] & sB \end{tikzcd}$$

shows that diagram (2.16) commutes on generators. Therefore, it is commutative. □

Reducing the data used in Section 0.28 or Definition 1.19 we call an *n-dimensional right operad module* the pair $(\mathcal{P};\mathcal{B})$ consisting of a $\mathbf{dg}$-operad $\mathcal{B}$ and an object $\mathcal{P} \in \mathbf{dg}^{\mathbb{N}^n}$, equipped with a unital associative action

$$\rho : \Big(\bigotimes_{q=1}^{l} \mathcal{P}(k_q)\Big) \otimes \mathcal{B}(l) \longrightarrow \mathcal{P}\Big(\sum_{q=1}^{l} k_q\Big) \in \mathbf{dg}\,.$$

2.12 Definition. An *n-dimensional right operad module homomorphism* $(h;g) : (\mathcal{P};\mathcal{B}) \to (\mathcal{Q};\mathcal{D})$ of degree $(p;0)$, $p \in \mathbb{Z}^n$, is a $\mathbf{dg}$-operad homomorphism $g : \mathcal{B} \to \mathcal{D}$ of degree 0 and a collection of homogeneous $\Bbbk$-linear maps $h(j) : \mathcal{P}(j) \to \mathcal{Q}(j)$, $j \in \mathbb{N}^n$, of degree $(p|j) = \sum_{i=1}^n p^i j^i$ such that

- for all $l \in \mathbb{N}$, $(k_q \in \mathbb{N}^n \mid 1 \leqslant q \leqslant l)$, the following square commutes up to the sign

$$\begin{tikzcd} \Big(\bigotimes_{q=1}^{l} \mathcal{P}(k_q)\Big) \otimes \mathcal{B}(l) \arrow[r, "\rho"] \arrow[d, "(\otimes_{q=1}^l h(k_q))\otimes g(l)"'] \arrow[dr, phantom, "(-1)^{c(k_1,\dots,k_l)}"] & \mathcal{P}\Big(\sum_{q=1}^{l} k_q\Big) \arrow[d, "h(\sum_{q=1}^l k_q)"] \\ \Big(\bigotimes_{q=1}^{l} \mathcal{Q}(k_q)\Big) \otimes \mathcal{D}(l) \arrow[r, "\rho"] & \mathcal{Q}\Big(\sum_{q=1}^{l} k_q\Big) \end{tikzcd} \quad (2.18)$$

$$c(k_1,\dots,k_l) = \sum_{1\leqslant t<q\leqslant l} \chi(k_t,k_q), \quad (2.19)$$

where $\chi : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{Z}/2$ is an arbitrary bilinear form (it is specified by a matrix $\chi \in \mathrm{Mat}(n,\mathbb{Z}/2)$);

- for all $j \in \mathbb{N}^n$
$$d \cdot h(j) = (-1)^{(p|j)} h(j) \cdot d : \mathcal{P}(j) \to \mathcal{Q}(j). \tag{2.20}$$

2.13 Lemma. *Let $(\mathcal{P};\mathcal{B})$ be an n-dimensional right* **dg***-operad module, let $g : \mathcal{B} \to \mathcal{D}$ be a* **dg***-operad isomorphism of degree 0. Let $h(j) : \mathcal{P}(j) \to \mathcal{Q}(j)$, $j \in \mathbb{N}^n$, be a collection of invertible homogeneous $\Bbbk$-linear maps of degree $(p|j)$ for some $p \in \mathbb{Z}^n$. Let $\chi : \mathbb{N}^n \times \mathbb{N}^n \to \mathbb{Z}/2$ be a bilinear form. Then $\mathcal{Q}$ admits a unique structure of an n-dimensional right $\mathcal{D}$-module such that $(h;g) : (\mathcal{P};\mathcal{B}) \to (\mathcal{Q};\mathcal{D})$ is a homomorphism of degree $(p;0)$ with respect to χ.*

Proof. The value of the differential in $\mathcal{Q}$ is fixed by (2.20). The unique candidate ρ for action of $\mathcal{D}$ on $\mathcal{Q}$ is found from diagram (2.18). This ρ is a chain map, as follows from a cubical diagram consisting of two faces (2.18) joined by differentials. Opposite faces of the cube commute up to the same sign, since $(p|\sum_{q=1}^{l} k_q) = \sum_{q=1}^{l}(p|k_q)$. Therefore, the both squares expressing commutation of ρ with the differential commute simultaneously.

Associativity of the action of $\mathcal{D}$ on $\mathcal{Q}$ is expressed by the pentagon

$$\begin{tikzcd}
\bigotimes_{q=1}^{l}\Big(\bigotimes_{t=1}^{n_q} \mathcal{Q}({}_tk_q) \otimes \mathcal{D}(n_q)\Big) \otimes \mathcal{D}(l) \arrow[r, "\otimes_{q=1}^{l}\rho\otimes 1"] \arrow[d, "\wr"] & \Big(\bigotimes_{q=1}^{l} \mathcal{Q}\Big(\sum_{t=1}^{n_q} {}_tk_q\Big)\Big) \otimes \mathcal{D}(l) \arrow[r, "\rho"] & \mathcal{Q}\Big(\sum_{q=1}^{l}\sum_{t=1}^{n_q} {}_tk_q\Big) \\
\Big(\bigotimes_{q=1}^{l}\bigotimes_{t=1}^{n_q} \mathcal{Q}({}_tk_q)\Big) \otimes \Big(\bigotimes_{q=1}^{l} \mathcal{D}(n_q)\Big) \otimes \mathcal{D}(l) \arrow[r, "1\otimes\mu_{\mathcal{D}}"] & \Big(\bigotimes_{q=1}^{l}\bigotimes_{t=1}^{n_q} \mathcal{Q}({}_tk_q)\Big) \otimes \mathcal{D}\Big(\sum_{q=1}^{l} n_q\Big) \arrow[ur, "\rho"] &
\end{tikzcd}$$

lying at the bottom of a rectangular prism, whose top face is the pentagon, expressing associativity of the action of $\mathcal{B}$ on $\mathcal{P}$. Vertical maps are tensor products of h and g. The walls commute up to sign. The product of these signs is $+1$, since

$$c\Big(\big(({}_tk_q)_{t=1}^{n_q}\big)_{q=1}^{l}\Big) = c\Big(\Big(\sum_{t=1}^{n_q} {}_tk_q\Big)_{q=1}^{l}\Big) + \sum_{q=1}^{l} c\big(({}_tk_q)_{t=1}^{n_q}\big)$$

due to definition (2.19) of c and bilinearity of χ.

Unitality of the action of $\mathcal{D}$ on $\mathcal{Q}$ follows from that for $\mathcal{B}$ and $\mathcal{P}$, since $c(k) = 0$, $k \in \mathbb{N}^n$. □

Cofibrant replacement of an $n \wedge 1$-operad module $(\mathcal{O},\mathcal{P}) \overset{\text{def}}{=} (\mathcal{O},\dots,\mathcal{O};\mathcal{P};\mathcal{O})$ is a trivial fibration $(\mathcal{A},\mathcal{F}) \to (\mathcal{O},\mathcal{P})$ such that the only map from the initial $n \wedge 1$-operad module $(\mathbb{1},0) \to (\mathcal{A},\mathcal{F})$ is a cofibration in ${}_n\mathrm{Op}_1$.

2.14 Theorem. *The $n\wedge 1$-operad module $(\mathrm{A}_\infty,\mathrm{F}_n)$ is a cofibrant replacement of (As, FAs_n). Moreover, $(\mathrm{A}_\infty,\mathrm{F}_n) \to (As, FAs_n)$ is a homotopy isomorphism in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$.*

Proof. Generate a free $n \wedge 1$-operad As-module $\overline{\mathrm{F}}_n$ by elements $\mathsf{f}_{j^1,\dots,j^n} \in \overline{\mathrm{F}}_n(j^1,\dots,j^n)$, $(j^1,\dots,j^n) \in \mathbb{N}^n - 0$, of degree $1 - j^1 - \dots - j^n$. Actually, $(As, \overline{\mathrm{F}}_n)$ is the coequalizer in ${}_n\mathrm{Op}_1$ of the pair of morphisms of collections

$$0, \mathrm{in} : (\Bbbk\{(m_2 \otimes 1)m_2 - (1 \otimes m_2)m_2, m_n \mid n \geqslant 3\}, 0) \rightrightarrows (\mathrm{A}_\infty, \mathrm{F}_n),$$

the second arrow is just the embedding. Therefore, the differential in $\overline{\mathrm{F}}_n$ reduces to

$$\mathsf{f}_\ell \partial = \sum_{q=1}^{n} \sum_{r+2+t=\ell^q} (-1)^{1+t+\ell^{q+1}+\dots+\ell^n} \lambda^q({}^r 1, m, {}^t 1; \mathsf{f}_{\ell - e_q}) - \sum_{q+r=\ell}^{q,r\in\mathbb{N}^n-0} (-1)^{|r|+\sum_{c>d} q^c r^d} \rho(\mathsf{f}_q, \mathsf{f}_r; m),$$

$m = m_2$, and the equation $\partial^2 = 0$ follows. Notice that the quadratic part of the differential

$$\mathsf{f}_\ell \bar{\partial} = \sum_{q+r=\ell}^{q,r\in\mathbb{N}^n-0} (-1)^{1-|r|+\sum_{c>d} q^c r^d} \rho(\mathsf{f}_q, \mathsf{f}_r; m) \tag{2.21}$$

is a differential itself, $\bar{\partial}^2 = 0$.

The $n \wedge 1$-operad module morphism in question decomposes as

$$(\mathrm{A}_\infty, \mathrm{F}_n) \xrightarrow{htis} (As, \overline{\mathrm{F}}_n) \xrightarrow{(1,p)} (As, FAs_n).$$

The first epimorphism is a homotopy isomorphism, since $\mathrm{A}_\infty \to As$ is. Let us describe the second epimorphism and prove for it the same property, that is, the $n \wedge 1$-operad As-module epimorphism $p : \overline{\mathrm{F}}_n \to FAs_n$ is a homotopy isomorphism. We prove more: the zero degree cycle $p : \overline{\mathrm{F}}_n \to FAs_n$ is homotopy invertible in the **dg**-category n-As-mod.

Any left n-operad As-module $\mathcal{P}$ decomposes into a direct sum of submodules. Any subset $S \subset \mathbf{n}$ with the induced total ordering is viewed as the isomorphic ordinal with $|S|$ elements. For any $k \in \mathbb{N}^n$ denote by $\operatorname{supp} k = \{i \in \mathbf{n} \mid k^i \neq 0\}$ its support. Consider the n-operad As-submodule

$$\mathcal{P}^S(k) = \begin{cases} \mathcal{P}(k) & \text{if } \operatorname{supp} k = S, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{P} = \oplus_{S\subset\mathbf{n}} \mathcal{P}^S$. Since $As(0) = 0$, the $\mathbb{Z}^n$-graded collection $\mathcal{P}^S$ is a left n-operad As-module. This structure boils down to a $\mathbb{Z}^S$-graded collection $\mathcal{P}^S$, which is a left S-operad As-module (that is, a $|S|$-operad As-module). A left n-operad As-module $\mathcal{P}$ is freely generated iff left S-operad As-modules $\mathcal{P}^S$ are freely generated for all $S \subset \mathbf{n}$.

Let $e_S \in \mathbb{N}^n$ have the coordinates $e_S^i = \chi(i \in S) \in \{0, 1\}$, $e_i \overset{\text{def}}{=} e_{\{i\}}$. For $j \in \mathbb{N}^n$, $j \neq 0$, consider the basic element $u_j = 1 \in \Bbbk = FAs_n(j)$. For $S \neq \varnothing$ the element $u_{e_S} = 1 \in \Bbbk = FAs_n(e_S)$ freely generates the left S-operad As-module FAs_n^S, while $FAs_n^\varnothing = 0$. Namely, for any $j \in \mathbb{N}^n$, $j \neq 0$, with support $S = \operatorname{supp} j$ we have $u_j = \lambda((m^{(j^i)})_{i\in S}; u_{e_S})$.

The left n-operad As-module $\overline{\mathrm{F}}_n$ is also freely generated. Its basis is given by elements $\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)})$, where $k > 0$ and $j_t \in \mathbb{N}^n - 0$ for all t.

The $n \wedge 1$-operad As-module map p is specified on the generators as follows:

$$\mathsf{f}_j.p = \begin{cases} u_j, & \text{if } |j| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

On the basis of the left n-operad As-module $\overline{\mathrm{F}}_n$ the map p is computed as

$$\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).p = \begin{cases} u_j, & \text{if} \quad |j_1| = \dots = |j_k| = 1, \quad j = \sum_r j_r, \\ 0, & \text{otherwise.} \end{cases}$$

In order to prove that p is a chain map it suffices to prove that $\mathsf{f}_{2e_a}.\partial p = 0$, $1 \leqslant a \leqslant n$, and $\mathsf{f}_{e_a+e_b}.\partial p = 0$ for all $1 \leqslant a < b \leqslant n$. These equations are verified straightforwardly:

$$\mathsf{f}_{2e_a}.\partial = -\lambda(m; \mathsf{f}_{e_a}) + \rho(\mathsf{f}_{e_a}, \mathsf{f}_{e_a}; m) \xmapsto{p} -\lambda(m; u_{e_a}) + u_{2e_a} = 0,$$
$$\mathsf{f}_{e_a+e_b}.\partial = \rho(\mathsf{f}_{e_a}, \mathsf{f}_{e_b}; m) - \rho(\mathsf{f}_{e_b}, \mathsf{f}_{e_a}; m) \xmapsto{p} u_{e_a+e_b} - u_{e_b+e_a} = 0.$$

A zero degree cycle $\beta : FAs_n \to \overline{\mathrm{F}}_n$ in n-As-mod is given on generators u_j of free $\Bbbk$-modules $FAs_n(j)$ by the formula

$$u_j.\beta = \rho\big((\lambda(m^{(j^i)}; \mathsf{f}_{e_i}))_{i \in \operatorname{supp} j}; m^{(|\operatorname{supp} j|)}\big).$$

The composition

$$\begin{array}{ccccc} FAs_n & \xrightarrow{\quad\beta\quad} & \overline{\mathrm{F}}_n & \xrightarrow{\quad p \quad} & FAs_n \\ u_{e_S} & \longmapsto & \rho((\mathsf{f}_{e_i})_{i \in S}; m^{(|S|)}) = (\otimes_{i \in S} \mathsf{f}_{e_i}) m^{(S)} & \longmapsto & (\otimes_{i \in S} u_{e_i}) m^{(S)} = u_{e_S} \end{array}$$

is the identity map. Let us prove that $p\beta$ is homotopy invertible. These two statements would imply that p is homotopy invertible in n-As-mod and β is its homotopy inverse.

Let $\overline{\mathrm{F}}_n^{(q)}$ be a n-As-submodule generated by $\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)})$, $k \leqslant q$, $\overline{\mathrm{F}}_n^{(0)} = 0$. This filtration induces the graded n-As-module with the components $\overline{\mathrm{F}}_n^{\{k\}} = \overline{\mathrm{F}}_n^{(k)} / \overline{\mathrm{F}}_n^{(k-1)}$. Since the differential in As vanishes, the differential $\partial : \overline{\mathrm{F}}_n^{(q)} \to \overline{\mathrm{F}}_n^{(q+1)}$ is a left n-operad As-module map. We look for a left n-operad As-module map $h : \overline{\mathrm{F}}_n \to \overline{\mathrm{F}}_n$ of degree -1 such that $\overline{\mathrm{F}}_n^{(q)}.h \subset \overline{\mathrm{F}}_n^{(q-1)}$. Consider the zero degree cycle

$$N = 1 - p\beta + h\partial + \partial h \overline{\mathrm{F}}_n \to \overline{\mathrm{F}}_n.$$

It satisfies $\overline{\mathrm{F}}_n^{(q)}.N \subset \overline{\mathrm{F}}_n^{(q)}$. We are going to choose h in such a way that N be locally nilpotent. Thus, $1 - N$ is invertible with the (well–defined) inverse $\sum_{a=0}^{\infty} N^a$. Therefore,

$$p\beta = 1 - N + h\partial + \partial h : \overline{\mathrm{F}}_n \to \overline{\mathrm{F}}_n$$

is homotopy invertible.

Since $\rho\big(\overline{\mathrm{F}}_n^{(q_1)}(j_1)\otimes\cdots\otimes\overline{\mathrm{F}}_n^{(q_k)}(j_k)\otimes As(k)\big)\subset\overline{\mathrm{F}}_n^{(q_1+\cdots+q_k)}(j_1+\cdots+j_k)$ there is an induced map between quotients:

$$\overline{\rho}:\overline{\mathrm{F}}_n^{\{q_1\}}(j_1)\otimes\cdots\otimes\overline{\mathrm{F}}_n^{\{q_k\}}(j_k)\otimes As(k)\to\overline{\mathrm{F}}_n^{\{q_1+\cdots+q_k\}}(j_1+\cdots+j_k)$$

The actions $\overline{\rho}$ assemble to an action of As on the sum $\overline{\mathrm{F}}_n^{\{\}}=\oplus_{q=0}^{\infty}\overline{\mathrm{F}}_n^{\{q\}}$. The quadratic differential $\bar\partial:\overline{\mathrm{F}}_n^{\{q\}}(j)\to\overline{\mathrm{F}}_n^{\{q+1\}}(j)$ from (2.21) induces a differential $\bar\partial$ in $\overline{\mathrm{F}}_n^{\{\}}$, thereby making it into a differential graded $n\wedge 1$-As-module. As a left n-As-module it is generated by its n-dimensional right As-**dg**-submodule $\bar{\mathsf{f}}_n^{\{\}}$:

$$\bar{\mathsf{f}}_n^{\{\}}=\oplus_{k=0}^{\infty}\bar{\mathsf{f}}_n^{\{k\}},\qquad \bar{\mathsf{f}}_n^{\{k\}}(j)=\Bbbk\big\{\rho(\mathsf{f}_{j_1},\dots,\mathsf{f}_{j_k};m^{(k)})\mid j_1+\cdots+j_k=j,\ \forall\, q\leqslant k\ j_q\in\mathbb{N}^n-0\big\}.$$

The matrix coefficients of $\bar\partial:\bar{\mathsf{f}}_n^{\{k\}}(j)\to\bar{\mathsf{f}}_n^{\{k+1\}}(j)$ are integers and we shall find $h:\bar{\mathsf{f}}_n^{\{k\}}(j)\to\bar{\mathsf{f}}_n^{\{k-1\}}(j)$ with the same property. Thus, instead of working over a general ring $\Bbbk$ we can assume that $\Bbbk=\mathbb{Z}$, and we do it till the end of the proof. Any such map h extends to a morphism of left n-As-modules in a unique way.

The operator induced by N in the graded n-As-module $\overline{\mathrm{F}}_n^{\{\}}$ is denoted $\bar N:\overline{\mathrm{F}}_n^{\{\}}\to\overline{\mathrm{F}}_n^{\{\}}$. It can be described via a simplified formula

$$\bar N=1-\overline{p\beta}+h\bar\partial+\bar\partial h:\overline{\mathrm{F}}_n^{\{k\}}\to\overline{\mathrm{F}}_n^{\{k\}},\tag{2.22}$$

where $\bar\partial$ is given by (2.21), $h:\overline{\mathrm{F}}_n^{\{p\}}\to\overline{\mathrm{F}}_n^{\{p-1\}}$, and $\rho(\mathsf{f}_{j_1},\dots,\mathsf{f}_{j_k};m^{(k)}).\overline{p\beta}$ vanishes unless $|j_1|=\cdots=|j_k|=1$ and $\operatorname{supp} j_q$ are all distinct for $1\leqslant q\leqslant k$. When $(j_1,\dots,j_k)$ is a permutation of $(e_{a_1},\dots,e_{a_k})$ with $1\leqslant a_1<\cdots<a_k\leqslant n$, then

$$\rho(\mathsf{f}_{j_1},\dots,\mathsf{f}_{j_k};m^{(k)}).\overline{p\beta}=\rho(\mathsf{f}_{e_{a_1}},\dots,\mathsf{f}_{e_{a_k}};m^{(k)}).$$

Otherwise, $\rho(\mathsf{f}_{j_1},\dots,\mathsf{f}_{j_k};m^{(k)}).\overline{p\beta}$ vanishes. The operator N is locally nilpotent iff $\bar N$ is. We shall achieve $\bar N=0$.

Let us define a family of graded abelian groups $\tilde f_n(j)$, $j\in\mathbb{N}^n$,

$$\tilde f_n(j)^k=\mathbb{Z}\big\{x(j_1,\dots,j_k)\mid j_q\in\mathbb{N}^n-0,\ j_1+\cdots+j_k=j\big\}.$$

The family $\tilde f_n$ has an obvious structure of a graded n-dimensional right As-module, namely,

$$\tilde\rho\big(x\big(({}_tj_1)_{t=1}^{n_1}\big),\dots,x\big(({}_tj_k)_{t=1}^{n_k}\big);m^{(k)}\big)=x\big(({}_tj_1)_{t=1}^{n_1}),\dots,({}_tj_k)_{t=1}^{n_k}\big).$$

This structure is completely fixed by the requirement

$$\tilde\rho\big(x(j_1),\dots,x(j_k);m^{(k)}\big)=x(j_1,\dots,j_k)\tag{2.23}$$

Consider the bilinear form $\chi:\mathbb{N}^n\times\mathbb{N}^n\to\mathbb{Z}$, $\chi(t,p)=\sum_{c\leqslant d}t^cp^d$, and define the corresponding c by (2.19). There are invertible mappings $\psi(j):\bar{\mathsf{f}}_n^{\{\}}(j)\to\tilde f_n(j)$ of degree $|j|=((1,1,\dots,1)|j)$ such that

- $(\mathsf{f}_j).\psi(j) = x(j)$;
- the right As-module structure obtained from (ψ, id_{As}) and the bilinear form χ as in Lemma 2.13 satisfies condition (2.23).

Existence and uniqueness of ψ is shown in the following computation in square (2.18):

$$\begin{tikzcd}
\mathsf{f}_{j_1} \otimes \cdots \otimes \mathsf{f}_{j_k} \otimes m^{(k)} \arrow[r, mapsto, "\overline{\rho}"] \arrow[d, mapsto, "\psi(j_1)\otimes\cdots\otimes\psi(j_k)\otimes 1"'] & \rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}) \arrow[r, mapsto, "\psi(j_1+\cdots+j_k)"] & \rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).\psi \arrow[d, mapsto, "(-1)^{\sum_{q<r}^{c\leqslant d} j_q^c j_r^d}"] \\
(-1)^{\sum_{q<r} |j_q|(1-|j_r|)} x(j_1) \otimes \cdots \otimes x(j_k) \otimes m^{(k)} \arrow[rr, mapsto, "\tilde{\rho}"] & & (-1)^{\sum_{q=1}^k (k-q)|j_q| - \sum_{q<r} |j_q|\cdot|j_r|} x(j_1, \dots, j_k)
\end{tikzcd}$$

wherefore

$$\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).\psi = (-1)^{\sum_{q=1}^k (k-q)|j_q| - \sum_{q<r}^{c>d} j_q^c j_r^d} x(j_1, \dots, j_k)$$

and $\deg \psi(j) = |j|$ as claimed. We conclude that for this ψ and χ the induced (by Lemma 2.13) right action of As on $\tilde{f}_n$ is the natural one.

Let us compute the differential $\tilde{\partial}$ in $\tilde{f}_n$. For $\ell = j_1 + \cdots + j_k$ the expression

$$\begin{aligned}
&(-1)^{|\ell|} \rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).\bar{\partial}\psi \\
&= \sum_{q=1}^{k} \sum_{t+p=j_q}^{t,p\neq 0} (-1)^{\sum_{r=q+1}^k (1-|j_r|) + \sum_{c>d} t^c p^d + |p| + 1 + |\ell|} \rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_{q-1}}, \mathsf{f}_t, \mathsf{f}_p, \mathsf{f}_{j_{q+1}}, \dots, \mathsf{f}_{j_k}; m^{(k+1)}).\psi \\
&= \sum_{q=1}^{k} \sum_{t+p=j_q}^{t,p\neq 0} (-1)^{\sum_{r=q+1}^k (1-|j_r|) + \sum_{c>d} t^c p^d + |p| + 1 + |\ell| + \sum_{r=1}^{q-1} (k+1-r)|j_r| + (k+1-q)|t| + (k-q)|p|} \\
&\qquad \times (-1)^{\sum_{r=q+1}^k (k-r)|j_r| - \sum_{u<r}^{c>d} j_u^c j_r^d + \sum_{c>d} t^c p^d} x(j_1, \dots, j_{q-1}, t, p, j_{q+1}, \dots, j_k)
\end{aligned}$$

has to coincide with

$$\rho(\mathsf{f}_{j_1}, \dots, \mathsf{f}_{j_k}; m^{(k)}).\psi\tilde{\partial} = (-1)^{\sum_{q=1}^k (k-q)|j_q| - \sum_{q<r}^{c>d} j_q^c j_r^d} x(j_1, \dots, j_k).\tilde{\partial}.$$

This gives the differential $\tilde{\partial}$:

$$x(j_1, \dots, j_k).\tilde{\partial} = \sum_{q=1}^{k} (-1)^{k+1-q} \sum_{t+p=j_q}^{t,p\neq 0} x(j_1, \dots, j_{q-1}, t, p, j_{q+1}, \dots, j_k). \tag{2.24}$$

Note that the differential $\bar{\partial} : \bar{\mathsf{f}}_n^{\{k\}}(\ell) \to \bar{\mathsf{f}}_n^{\{k+1\}}(\ell)$ makes

$$0 \longrightarrow \bar{\mathsf{f}}_n^{\{1\}}(\ell) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \bar{\mathsf{f}}_n^{\{k\}}(\ell) \xrightarrow{\bar{\partial}} \dots \xrightarrow{\bar{\partial}} \bar{\mathsf{f}}_n^{\{|\ell|\}}(\ell) \longrightarrow 0 \tag{2.25}$$

into a bounded complex of abelian groups. The term $\bar{\mathsf{f}}_n^{\{k\}}(\ell)$ is placed in degree $k - |\ell|$.

Consider the operad morphism $As \to \mathbb{Z}$, $m^{(k)} \mapsto 0$ for $k \geqslant 2$, where $\mathbb{Z}$ is the unit operad, $\mathbb{Z}(1) = \mathbb{Z}$, $\mathbb{Z}(n) = 0$ for $n \neq 1$. We may view $\oplus_{\ell \in \mathbb{N}^n - 0} \bar{\mathsf{f}}_n^{\{k\}}(\ell)$ as a left n-operad $\mathbb{Z}$-module, quotient of $\bar{\mathsf{f}}_n^{\{k\}}$ by the submodule spanned by images of all left actions of elements $m^{(k)}$ for $k \geqslant 2$. Applying the same quotient procedure to FAs_n we get

$$\overline{FAs}_n(\ell) = \begin{cases} \mathbb{Z} = \mathbb{Z}u(\ell) = \mathbb{Z}u(e_{\operatorname{supp}\ell}), & \text{if } |\ell| = |\operatorname{supp}\ell|, \\ 0, & \text{if } |\ell| > |\operatorname{supp}\ell|. \end{cases}$$

We are going to prove that complex (2.25) is homotopy isomorphic via $\bar{p}$ and $\bar{\beta}$ to its cohomology $\overline{FAs}_n(\ell)$. If $\ell = e_S$ for some $S \subset \mathbf{n}$, then the cohomology is concentrated in degree 0 and equals $\overline{FAs}_n(e_S) = \mathbb{Z} = \mathbb{Z}u(e_S)$. If $|\ell| > |\operatorname{supp}\ell|$, then the cohomology vanishes.

We construct mappings of abelian groups $h : \bar{\mathsf{f}}_n^{\{p\}}(\ell) \to \bar{\mathsf{f}}_n^{\{p-1\}}(\ell)$ such that $\bar{N} : \bar{\mathsf{f}}_n^{\{k\}}(\ell) \to \bar{\mathsf{f}}_n^{\{k\}}(\ell)$ given by (2.22) vanishes. These h induce the left n-As-module morphism $h : \overline{\mathrm{F}}_n \to \overline{\mathrm{F}}_n$ compatibly with the generator–to–generator mapping $\bar{\mathsf{f}}_n^{\{k\}} \to \overline{\mathrm{F}}_n^{(k)}$. Thus, vanishing of $\bar{N} : \bar{\mathsf{f}}_n^{\{k\}}(\ell) \to \bar{\mathsf{f}}_n^{\{k\}}(\ell)$ implies vanishing of $\bar{N} : \overline{\mathrm{F}}_n^{\{\}} \to \overline{\mathrm{F}}_n^{\{\}}$ and local nilpotency of $N : \overline{\mathrm{F}}_n \to \overline{\mathrm{F}}_n$.

We have reduced the proposition to proving that the chain maps $\bar{p}$, $\bar{\beta}$ in

$$\begin{tikzcd} 0 \arrow[r] & \bar{\mathsf{f}}_n^{\{1\}}(\ell) \arrow[r, "\bar{\partial}"] & \dots \arrow[r, "\bar{\partial}"] & \bar{\mathsf{f}}_n^{\{k\}}(\ell) \arrow[r, "\bar{\partial}"] & \dots \arrow[r, "\bar{\partial}"] & \bar{\mathsf{f}}_n^{\{|\ell|-1\}}(\ell) \arrow[r, "\bar{\partial}"] & \bar{\mathsf{f}}_n^{\{|\ell|\}}(\ell) \arrow[r] \arrow[d, shift left, "\bar{p}"] & 0 \\ & & & & & 0 \arrow[r] & \overline{FAs}_n(\ell) \arrow[r] \arrow[u, shift left, "\bar{\beta}"] & 0 \end{tikzcd} \tag{2.26}$$

are homotopy inverse to each other for any $\ell \in \mathbb{N}^n - 0$. We add formally the case of $\ell = 0$ by defining the top and the bottom rows as complexes $\bar{\mathsf{f}}_n^{\{0\}}(0) = \mathbb{Z}$ and $\overline{FAs}_n(0) = \mathbb{Z}$ concentrated in degree 0. Here $\bar{p}$, $\bar{\beta}$ are defined as the identity maps.

Chain maps $\bar{p}$, $\bar{\beta}$ give rise to other chain maps $\tilde{p}$, $\tilde{\beta}$ in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \bar{\mathsf{f}}_n^{\{1\}}(\ell) \arrow[r, "\bar{\partial}"] \arrow[dd, "\psi"] & \dots \arrow[r, "\bar{\partial}"] & \bar{\mathsf{f}}_n^{\{k\}}(\ell) \arrow[r, "\bar{\partial}"] \arrow[dd, "\psi"] & \dots \arrow[r, "\bar{\partial}"] & \bar{\mathsf{f}}_n^{\{|\ell|\}}(\ell) \arrow[r] \arrow[dd, "\psi"] & 0 \arrow[d, shift left, "\bar{p}"] & & \\ & & & & & & 0 \arrow[r] \arrow[u, shift left, "\bar{\beta}"] & \overline{FAs}_n(\ell) \arrow[r] \arrow[dd, equal] & 0 \\ 0 \arrow[r] & \tilde{f}_n(\ell)^1 \arrow[r, "\tilde{\partial}"] & \dots \arrow[r, "\tilde{\partial}"] & \tilde{f}_n(\ell)^k \arrow[r, "\tilde{\partial}"] & \dots \arrow[r, "\tilde{\partial}"] & \tilde{f}_n(\ell)^{|\ell|} \arrow[r] & 0 \arrow[d, shift left, "\tilde{p}"] & & \\ & & & & & & 0 \arrow[r] \arrow[u, shift left, "\tilde{\beta}"] & \overline{FAs}_n(\ell) \arrow[r] & 0 \end{tikzcd}$$

In fact, the maps $\tilde{p}$, $\tilde{\beta}$ have to be defined if $l^i \in \{0, 1\}$ for all $1 \leqslant i \leqslant n$. For $k = |\ell|$ we find

$$x(e_{a_1}, \dots, e_{a_k}).\tilde{p} = \operatorname{sign}(a_1, \dots, a_k)u(\ell) \overset{\text{def}}{=} (-1)^{\sum_{q<r} \chi(a_q > a_r)} u(\ell), \tag{2.27}$$

where $\chi(b > c)$ is 1 or 0 depending on the case whether the inequality holds or not. The exponent is the number of inversions in the sequence $(a_1, \dots, a_k)$. If $\ell = e_{\mathbf{n}} = (1, 1, \dots, 1)$,

then $k = n$ and the sign is just the sign of the permutation $(a_1, \dots, a_k)$. The map $\tilde\beta = \bar\beta\psi : \overline{FAs}_n(\ell) \to \tilde f_n(\ell)^{|\ell|}$ satisfies

$$u(\ell).\tilde\beta = x(e_{c_1}, \dots, e_{c_k}), \tag{2.28}$$

where $\{c_1 < c_2 < \dots < c_k\} = \operatorname{supp}\ell$. In particular, if $\ell = e_{\mathbf{n}}$, then $k = n$ and $u(e_{\mathbf{n}})\tilde\beta = x(e_1, \dots, e_n)$.

Consider the augmented coalgebra $C_n = \mathbb{Z}\{\mathbb{N}^n\}$ with the comultiplication $j.\Delta = \sum_{q+r=j} q \otimes r$ where $j, q, r \in \mathbb{N}^n$. Generators $j \in \mathbb{N}^n$ of the free abelian group C_n are denoted also $x(j)$. The augmentation is $\eta : \mathbb{Z} \to C_n$, $1 \mapsto x(0)$. The counit is $\varepsilon : C_n \to \mathbb{Z}$, $x(j) \mapsto \delta_{j0}$. The reduced comultiplication is defined as

$$\bar\Delta = \Delta - \eta \otimes \mathrm{id} - \mathrm{id} \otimes \eta + \varepsilon\eta \otimes \eta, \qquad \vec{0}.\bar\Delta = 0, \quad j.\bar\Delta = \sum_{q+r=j}^{q,r\neq 0} q \otimes r \quad \text{for } j \neq 0.$$

The abelian subgroup $\bar C_n = \operatorname{Ker}\varepsilon = \mathbb{Z}\{\mathbb{N}^n - 0\} \subset C_n$ equipped with the comultiplication $\bar\Delta$ is a coassociative coalgebra, which is not counital. The complex $\tilde f_n$ is nothing else but the cohomology complex $K'(\bar C_n)$ of the coassociative coalgebra $\bar C_n$, which is the upper row of the diagram

$$\begin{tikzcd}[column sep=small]
0 \to \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \bar C_n \arrow[r, "\bar\Delta"] \arrow[d, shift left, "\tilde p"] & \bar C_n^{\otimes 2} \arrow[r, "\tilde\partial"] \arrow[d, shift left, "\tilde p"] & \dots \arrow[r, "\tilde\partial"] & \bar C_n^{\otimes k} \arrow[r, "\tilde\partial"] \arrow[d, shift left, "\tilde p"] & \dots \arrow[r, "\tilde\partial"] & \bar C_n^{\otimes n} \arrow[r, "\tilde\partial"] \arrow[d, shift left, "\tilde p"] & \dots \\
0 \to \mathbb{Z} \arrow[r, "0"] & \wedge^1_{\mathbb{Z}}(\mathbb{Z}^n) \arrow[r, "0"] \arrow[u, shift left, "\tilde\beta"] & \wedge^2_{\mathbb{Z}}(\mathbb{Z}^n) \arrow[r, "0"] \arrow[u, shift left, "\tilde\beta"] & \dots \arrow[r, "0"] & \wedge^k_{\mathbb{Z}}(\mathbb{Z}^n) \arrow[r, "0"] \arrow[u, shift left, "\tilde\beta"] & \dots \arrow[r, "0"] & \wedge^n_{\mathbb{Z}}(\mathbb{Z}^n) \arrow[r, "0"] \arrow[u, shift left, "\tilde\beta"] & 0
\end{tikzcd} \tag{2.29}$$

We identify $x(j_1, \dots, j_k) \in \tilde f_n$ with $j_1 \otimes \dots \otimes j_k \in \bar C_n^{\otimes k}$. The exterior algebra $\wedge_{\mathbb{Z}}(\mathbb{Z}^n) = T_{\mathbb{Z}}(\mathbb{Z}^n)/(x \otimes x \mid x \in \mathbb{Z}^n)$ has the basis $(e_{\{c_1<c_2<\dots<c_k\}} = e_{c_1} \wedge e_{c_2} \wedge \dots \wedge e_{c_k})$, where $1 \leqslant c_1 < c_2 < \dots < c_k \leqslant n$. The mappings in this diagram are

$$(j_1 \otimes \dots \otimes j_k).\tilde\partial = \sum_{q=1}^{k} (-1)^{k-q+1} j_1 \otimes \dots \otimes j_{q-1} \otimes j_q.\bar\Delta \otimes j_{q+1} \otimes \dots \otimes j_k,$$

$$x(j_1, \dots, j_k).\tilde p = 0 \;\text{ unless }\; \Big|\sum_{q=1}^{k} j_q\Big| = k = \Big|\operatorname{supp}\sum_{q=1}^{k} j_q\Big|, \quad j_q \in \mathbb{N}^n - 0,$$

$$x(e_{a_1}, \dots, e_{a_k}).\tilde p = (-1)^{\sum_{q<r}\chi(a_q>a_r)} e_{\{a_1,\dots,a_k\}}, \quad a_1, \dots, a_k - \text{distinct},$$

$$e_{\{c_1<c_2<\dots<c_k\}}.\tilde\beta = x(e_{c_1}, \dots, e_{c_k}),$$

which coincides with (2.24), (2.27) and (2.28).

It remains to prove that the maps $\tilde p$, $\tilde\beta$ are homotopy inverse to each other. Clearly, $\tilde\beta\tilde p = 1$.

2.15 Lemma. *For $n = 1$ the maps $\tilde p$, $\tilde\beta$ are homotopy inverse to each other.*

Proof. For $n = 1$ the chain maps in question become

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \bar{C}_1 \arrow[r, "\bar{\Delta}"] \arrow[d, shift left, "\tilde{p}"] & \bar{C}_1^{\otimes 2} \arrow[r, "\tilde{\partial}"] \arrow[d, shift left] & \dots \arrow[r, "\tilde{\partial}"] & \bar{C}_1^{\otimes k} \arrow[r, "\tilde{\partial}"] \arrow[d, shift left] & \dots \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r] \arrow[u, shift left, "\tilde{\beta}"] & 0 \arrow[r] \arrow[u, shift left] & \dots \arrow[r] & 0 \arrow[r] \arrow[u, shift left] & \dots
\end{tikzcd}$$

$$x(j).\tilde{p} = \delta_{j1}, \qquad 1.\tilde{\beta} = x(1).$$

Define a map of graded abelian groups $h : K'(\bar{C}_1) \to K'(\bar{C}_1)$ of degree -1 by the formula

$$x(j_1, \dots, j_{k-1}, j_k).h = \begin{cases} x(j_1, \dots, j_{k-2}, j_{k-1} + 1), & \text{if } k > 1,\ j_k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We claim that the chain map $E = \tilde{p}\tilde{\beta} - h\tilde{\partial} - \tilde{\partial}h : K'(\bar{C}_1) \to K'(\bar{C}_1)$ is the identity map. In fact, $x(1).E = x(1)$, and for $j \geqslant 2$ we have

$$x(j).E = \sum_{q+r=j}^{q,r>0} x(q, r).h = x(j - 1 + 1) = x(j).$$

For $k > 1$ and $j_k \geqslant 2$ we find

$$x(j_1, \dots, j_k).E = \sum_{q+r=j_k} x(j_1, \dots, j_{k-1}, q, r).h = x(j_1, \dots, j_{k-1}, j_k).$$

It remains to consider for $k > 1$ the value

$$\begin{aligned} &x(j_1, \dots, j_{k-1}, 1).E = -x(j_1, \dots, j_{k-2}, j_{k-1} + 1).\tilde{\partial} - x(j_1, \dots, j_{k-1}, 1).\tilde{\partial}h \\ &= \sum_{q+r=j_{k-1}+1} x(j_1, \dots, j_{k-1}, q, r) - \sum_{q+t=j_{k-1}} x(j_1, \dots, j_{k-2}, q, t, 1).h = x(j_1, \dots, j_{k-2}, j_{k-1}, 1). \end{aligned}$$

Hence, $E = \mathrm{id}$, and $\tilde{p}\tilde{\beta}$ is homotopic to the identity map. □

2.15.1. Homology of augmented algebras. Let $\mathcal{C}$ be a symmetric monoidal category with the tensor product $\otimes$ and the unit object $\mathbb{1}$. Assume that $A = (A, \mu : A \otimes A \to A, \eta : \mathbb{1} \to A, \varepsilon : A \to \mathbb{1})$ is an augmented unital associative algebra in $\mathcal{C}$. There is an associated simplicial object $S(A)$:

$$\begin{tikzcd}[column sep=huge]
\cdots A \otimes A \otimes A \arrow[r, shift left=6, "d_3=1\otimes 1\otimes\varepsilon"] \arrow[r, shift left=2, "d_2=1\otimes\mu"] \arrow[r, shift right=2, "d_1=\mu\otimes 1"] \arrow[r, shift right=6, "d_0=\varepsilon\otimes 1\otimes 1"'] & A \otimes A \arrow[l, shift right=4, "s_2=1\otimes 1\otimes\eta"'] \arrow[l, "s_1=1\otimes\eta\otimes 1"] \arrow[l, shift left=4, "s_0=\eta\otimes 1\otimes 1"] \arrow[r, shift left=4, "d_2=1\otimes\varepsilon"] \arrow[r, "d_1=\mu"] \arrow[r, shift right=4, "d_0=\varepsilon\otimes 1"'] & A \arrow[l, shift right=2, "s_1=1\otimes\eta"'] \arrow[l, shift left=2, "s_0=\eta\otimes 1"] \arrow[r, shift left=2, "d_1=\varepsilon"] \arrow[r, shift right=2, "d_0=\varepsilon"'] & \mathbb{1}, \arrow[l, "s_0=\eta"']
\end{tikzcd}$$

where d_i and s_i are face maps and degeneracy maps respectively. When B is another augmented algebra in $\mathcal{C}$, the Cartesian product $S(A) \times S(B)$ of simplicial objects [Mac63, Section VIII.8] is naturally isomorphic to the simplicial object $S(A \otimes B)$.

Assume also that $\mathcal{C}$ is abelian and the tensor product $\otimes$ is bilinear. A complex $K(A) \stackrel{\text{def}}{=} K(S(A))$ is associated with the simplicial object $S(A)$. It has the differential $\partial = \sum_{i=0}^{q}(-1)^i d_i : A^{\otimes q} \to A^{\otimes q-1}$. Homology of $K(A)$ gives the torsion objects $\operatorname{Tor}^A_\bullet(\mathbb{1}, \mathbb{1})$, where the left and the right A-module $\mathbb{1}$ obtains its structure via $\varepsilon : A \to \mathbb{1}$. Given two augmented algebras A and B in $\mathcal{C}$ we can form a bisimplicial object in $\mathcal{C}$, whose terms are $A^{\otimes p} \otimes B^{\otimes q}$. By Eilenberg–Zilber theorem [Wei94, Theorem 8.5.1] the complexes $K(A \otimes B) = K(S(A \otimes B)) \simeq K(S(A) \times S(B))$ and $K(A) \otimes K(B)$ are quasi-isomorphic.

Consider associative algebras $\bar{A} = (\bar{A}, \mu : \bar{A} \otimes \bar{A} \to \bar{A})$ in $\mathcal{C}$, which are not required to have a unit. Such an algebra gives rise to a unital one $A = \mathbb{1} \oplus \bar{A}$ for which $\eta = \operatorname{in}_{\mathbb{1}} : \mathbb{1} \to A$ is the unit and $\varepsilon = \operatorname{pr}_{\mathbb{1}} : A \to \mathbb{1}$ is an augmentation. Introduce another monoidal product in $\mathcal{C}$ (not bilinear) via the formula

$$\bar{A} \circledast \bar{B} = \bar{A} \oplus \bar{B} \oplus (\bar{A} \otimes \bar{B}).$$

There is an obvious isomorphism

$$\mathbb{1} \oplus (\bar{A} \circledast \bar{B}) = (\mathbb{1} \oplus \bar{A}) \otimes (\mathbb{1} \oplus \bar{B}).$$

If $\bar{A}$, $\bar{B}$ are associative algebras in $(\mathcal{C}, \otimes)$, then $\bar{A} \circledast \bar{B}$ obtains an associative algebra structure in $(\mathcal{C}, \otimes)$ via this isomorphism, namely, $\mathbb{1} \oplus (\bar{A} \circledast \bar{B}) = A \otimes B$.

There is a normalised chain complex $K_N(\bar{A}) = K_N(S(A))$ of the simplicial complex $S(A)$:

$$\dots \longrightarrow \bar{A}^{\otimes 3} \xrightarrow{-\mu \otimes 1 + 1 \otimes \mu} \bar{A}^{\otimes 2} \xrightarrow{-\mu} \bar{A} \xrightarrow{0} \mathbb{1} \to 0,$$

with the differential $\partial = \sum_{i=0}^{q-2}(-1)^{i+1} 1^{\otimes i} \otimes \mu \otimes 1^{\otimes q-i-2} : \bar{A}^{\otimes q} \to \bar{A}^{\otimes q-1}$, where $\mathbb{1}$ is placed in degree 0. By a generalization of normalization theorem of Eilenberg and Mac Lane [Mac63, Theorem VIII.6.1] the natural projection $K(A) \to K_N(\bar{A})$ is a homotopy isomorphism. As a corollary, we get the following

2.16 Proposition. *For associative algebras $\bar{A}$, $\bar{B}$ in $(\mathcal{C}, \otimes)$ there is a natural quasi-isomorphism*

$$K_N(\bar{A} \circledast \bar{B}) \rightleftarrows K_N(\bar{A}) \otimes K_N(\bar{B}).$$

2.16.1. Conclusion of the proof of Theorem 2.14. Let us take for $(\mathcal{C}, \otimes)$ the category $(\mathrm{Ab}^{\mathrm{op}}, \otimes^{\mathrm{op}}_{\mathbb{Z}})$ opposite to the category of abelian groups with the opposite tensor product. Clearly, the category $\mathrm{Ab}^{\mathrm{op}}$ is abelian. An associative algebra in this monoidal category is a coassociative coalgebra over $\mathbb{Z}$ in the ordinary sense. In particular, such is $\bar{C}_n = \mathbb{Z}\{\mathbb{N}^n - 0\}$. Adding a unit to it in $\mathrm{Ab}^{\mathrm{op}}$ gives $C_n = \mathbb{Z}\{\mathbb{N}^n\}$. Since $C_n \otimes_{\mathbb{Z}} C_m \simeq C_{n+m}$, we conclude that $\bar{C}_n \circledast \bar{C}_m \simeq \bar{C}_{n+m}$. The homological complex $K_N(\bar{C}_n)$ in $\mathrm{Ab}^{\mathrm{op}}$ and the top line of (2.29), the cohomological complex $K'(\bar{C}_n)$ in Ab are identified: the n-th abelian groups and the

differentials between them coincide. Thus, the results of the previous section apply to $K'(\bar{C}_n)$.

We claim that the cohomology of $K'(\bar{C}_n)$ is isomorphic to $\wedge_{\mathbb{Z}}(\mathbb{Z}^n)$. In fact, using induction we deduce from Lemma 2.15 and Proposition 2.16 the quasi-isomorphism

$$K'(\bar{C}_{n+1}) \xrightarrow[qis]{} K'(\bar{C}_n) \otimes K'(\bar{C}_1) \xrightarrow{q\otimes\tilde{p}} \wedge_{\mathbb{Z}} (\mathbb{Z}^n) \otimes \wedge_{\mathbb{Z}}(\mathbb{Z}) \simeq \wedge_{\mathbb{Z}}(\mathbb{Z}^{n+1}).$$

Here q, $\tilde{p}$ are quasi-isomorphisms. So is their tensor product $q\otimes\tilde{p}$, since complexes $K'(\bar{C}_n)$, $\wedge_{\mathbb{Z}}(\mathbb{Z}^n)$ consist of free abelian groups, $\wedge_{\mathbb{Z}}(\mathbb{Z}^n)$ are bounded and $K'(\bar{C}_n)$ are direct sums of bounded complexes.

Both rows of diagram (2.29) have the same homology, which coincides with the bottom row and consists of finitely generated free abelian groups. Since $H(\tilde{\beta})H(\tilde{p}) = 1$, the matrices of $H(\tilde{\beta})$ and $H(\tilde{p})$ are invertible. Thus, $\tilde{\beta}$ and $\tilde{p}$ induce isomorphisms in homology. They are quasi-isomorphisms of complexes consisting of free abelian groups. Therefore, their cones are acyclic complexes consisting of free abelian groups. They split into short exact sequences whose terms are also free abelian groups (as subgroups of such). Hence, these short exact sequences split and the cones are contractible. Thus, $\tilde{p}$ and $\tilde{\beta}$ are homotopy isomorphisms. Clearly, they are homotopy inverse to each other. This implies the same conclusion for $\bar{p}$ and $\bar{\beta}$ and for p and β. □

2.17 Corollary (to Proposition 2.10, Theorem 2.14)**.** *The polymodule F_n is homotopy isomorphic to its cohomology and $H^\bullet(F_n(j)) = \Bbbk[1 - |j|]$ for $j \in \mathbb{N}^n - 0$.*

This is due to existence of a degree 1 isomorphism $\Sigma : H^\bullet(F_n) \to FAs_n$.

2.18. Homotopy unital A_∞-morphisms. Consider the free $n \wedge 1$-$\mathrm{A}_\infty^{\mathsf{su}}$-module

$$\tilde{\mathrm{F}}_n = \bigodot_{i=1}^n \mathrm{A}_\infty^{\mathsf{su}} \odot^i_{\mathrm{A}_\infty} \mathrm{F}_n \odot^0_{\mathrm{A}_\infty} \mathrm{A}_\infty^{\mathsf{su}} = \odot_{\geqslant 0}({}^n\mathrm{A}_\infty^{\mathsf{su}}; \Bbbk\{\mathsf{f}_j \mid j \in \mathbb{N}^n - 0\}; \mathrm{A}_\infty^{\mathsf{su}}).$$

In particular, $\tilde{\mathrm{F}}_0 = \mathrm{A}_\infty^{\mathsf{su}}(0) = \Bbbk 1^{\mathsf{su}}$ by Lemma 1.34. The graded ideal generated by the following system of relations in it

$$\rho_\varnothing(1^{\mathsf{su}}) = \lambda^i_{e_i}(1^{\mathsf{su}}; \mathsf{f}_{e_i}),\ \forall\, i, \quad \lambda^i_\ell({}^a1, 1^{\mathsf{su}}, {}^b1; \mathsf{f}_\ell) = 0 \ \text{ if } \ a + 1 + b = \ell^i,\ |\ell| > 1,$$

is stable under the differential, as one easily verifies. Therefore the quotient $\mathrm{F}_n^{\mathsf{su}}$ of $\tilde{\mathrm{F}}_n$ by these relations is an $n \wedge 1$-$\mathrm{A}_\infty^{\mathsf{su}}$-module. We still have $\mathrm{F}_0^{\mathsf{su}} = \mathrm{A}_\infty^{\mathsf{su}}(0) = \Bbbk 1^{\mathsf{su}}$. Note that $\mathrm{F}_n^{\mathsf{su}}$-algebra maps coincide with *strictly unital* A_∞*-algebra morphisms*, which are by [BLM08, Definition 9.2] A_∞-morphisms $\mathsf{f} : (A_1, \dots, A_n) \to B$ between strictly unital A_∞-algebras such that all components of f vanish if any of its entries is $1^{\mathsf{su}}_{A_i}$, except $1^{\mathsf{su}}_{A_i}\mathsf{f}_{e_i} = 1^{\mathsf{su}}_B$.

The rows of the following diagram in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{A}_\infty, \mathrm{F}_n) \arrow[r] \arrow[d, two heads, "htis"'] & (\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}}) \arrow[r] \arrow[d, two heads, "htis"', "p'"] & (\Bbbk 1^{\mathsf{su}}, \Bbbk 1^{\mathsf{su}}\rho_\varnothing) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (As, FAs_n) \arrow[r] & (As1, FAs1_n) \arrow[r] & (\Bbbk 1^{\mathsf{su}}, \Bbbk 1^{\mathsf{su}}\rho_\varnothing) \arrow[r] & 0
\end{tikzcd} \tag{2.30}$$

are exact sequences, split in the obvious way. Therefore, the middle vertical arrow p' is a homotopy isomorphism.

Consider the embedding of free graded operads $\mathrm{A}_\infty^{\mathsf{su}} \to \mathrm{A}_\infty^{\mathsf{su}}\langle\mathsf{i},\mathsf{j}\rangle$, where i, j are two nullary operations, $\deg\mathsf{i} = 0$, $\deg\mathsf{j} = -1$. Assuming $\mathsf{i}\partial = 0$, $\mathsf{j}\partial = \mathbf{1}^{\mathsf{su}} - \mathsf{i}$, we make the second operad differential graded and the embedding becomes a chain map. It is proven in [Lyu11] (end of proof of Proposition 1.8) that this embedding is a homotopy isomorphism. Or, the reader can simplify the lines of the proof given below and adopt it to the case of $\mathrm{A}_\infty^{\mathsf{su}} \to \mathrm{A}_\infty^{\mathsf{su}}\langle\mathsf{i},\mathsf{j}\rangle$.

2.19 Proposition. *The embedding* $\iota : (\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}}) \to (\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i},\mathsf{j}\rangle$ *is a homotopy isomorphism.*

Proof. An arbitrary chain $n \wedge 1$-module map $\phi : (\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i},\mathsf{j}\rangle \to (\mathcal{A}, \mathcal{P})$ is fixed by specifying a chain $n \wedge 1$-module map $(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}}) \to (\mathcal{A}, \mathcal{P})$ and the image $\phi(\mathsf{j}) \in \mathcal{A}(0)^{-1}$. In particular, there is a unique chain $n \wedge 1$-module map

$$\pi : (\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i},\mathsf{j}\rangle \to (\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}}), \qquad \mathsf{i} \mapsto \mathbf{1}^{\mathsf{su}}, \qquad \mathsf{j} \mapsto 0,$$

whose restriction to $(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})$ is identity. Let us prove that π is homotopy inverse to ι.

The restrictions of the above chain maps $\iota' : \Bbbk\mathbf{1}^{\mathsf{su}} \hookrightarrow \Bbbk\{\mathbf{1}^{\mathsf{su}},\mathsf{i},\mathsf{j}\}$ and $\pi' : \Bbbk\{\mathbf{1}^{\mathsf{su}},\mathsf{i},\mathsf{j}\} \to \Bbbk\mathbf{1}^{\mathsf{su}}$, $\mathbf{1}^{\mathsf{su}} \mapsto \mathbf{1}^{\mathsf{su}}$, $\mathsf{i} \mapsto \mathbf{1}^{\mathsf{su}}$, $\mathsf{j} \mapsto 0$, are homotopy isomorphisms: the homotopy $h : \Bbbk\{\mathbf{1}^{\mathsf{su}},\mathsf{i},\mathsf{j}\} \to \Bbbk\{\mathbf{1}^{\mathsf{su}},\mathsf{i},\mathsf{j}\}$, $\mathbf{1}^{\mathsf{su}}.h = 0$, $\mathsf{i}.h = \mathsf{j}$, $\mathsf{j}.h = 0$, satisfies $\partial h + h\partial = \pi'\iota' - 1$. We know from Proposition 1.36 that the $n \wedge 1$-module $(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i},\mathsf{j}\rangle$ coincides with $(\mathrm{A}_\infty^{\mathsf{su}}\langle\mathsf{i},\mathsf{j}\rangle, \mathcal{P} = \bigodot_{k=0}^{n}\mathrm{A}_\infty^{\mathsf{su}}\langle\mathsf{i},\mathsf{j}\rangle \odot_{\mathrm{A}_\infty^{\mathsf{su}}}^{k} \mathrm{F}_n^{\mathsf{su}})$. The component $\mathcal{P}(l)$ is obtained from components $\mathrm{F}_n^{\mathsf{su}}(r)$ with $r^q \geqslant l^q$ for all $q \in \mathbf{n}$ by plugging the unused $r^q - l^q$ entries with i and j in all possible ways determined by injections $\psi^q : \mathbf{l}^q \hookrightarrow \mathbf{r}^q$:

$$\mathcal{P}(l) = \bigoplus_{(\psi^q:\mathbf{l}^q\hookrightarrow\mathbf{r}^q)_{q=1}^n} \Bigl(\bigotimes_{q\in\mathbf{n}} \bigotimes_{\mathbf{r}^q-\operatorname{Im}\psi^q} \Bbbk\{\mathsf{i},\mathsf{j}\}\Bigr) \otimes \mathrm{F}_n^{\mathsf{su}}(r).$$

There is a split surjection

$$\lambda : \mathcal{Q}(l) = \bigoplus_{(\psi^q:\mathbf{l}^q\hookrightarrow\mathbf{r}^q)_{q=1}^n} \Bigl(\bigotimes_{q\in\mathbf{n}} \bigotimes_{\mathbf{r}^q-\operatorname{Im}\psi^q} (\Bbbk\{\mathsf{i},\mathsf{j}\} \oplus \Bbbk\mathbf{1}^{\mathsf{su}})\Bigr) \otimes \mathrm{F}_n^{\mathsf{su}}(r) \longrightarrow \mathcal{P}(l),$$

obtained by acting with all $\mathbf{1}^{\mathsf{su}}$ on $\mathrm{F}_n^{\mathsf{su}}$ on the left via λ. This reduces the quantities r^q by the number of factors $\mathbf{1}^{\mathsf{su}}$.

Denote $f = \pi'\iota', g = \mathrm{id} : \Bbbk\{\mathbf{1}^{\mathsf{su}},\mathsf{i},\mathsf{j}\} \to \Bbbk\{\mathbf{1}^{\mathsf{su}},\mathsf{i},\mathsf{j}\}$. Equip the set $S = \sqcup_{q\in\mathbf{n}}(\mathbf{r}^q - \operatorname{Im}\psi^q)$ with the lexicographic order, $q < y \in \mathbf{n}$ implies $(q,c) < (y,z)$. The maps ∂ and $\pi\iota$ satisfy

$$\begin{tikzcd}[column sep=huge]
\mathcal{Q}(l) \arrow[r, "\oplus_{(\psi^q)_q}(1\otimes\partial+\sum_{(y,z)\in S}(\otimes^{(q,c)<(y,z)}1)\otimes\partial\otimes(\otimes^{(q,c)>(y,z)}1)\otimes 1)", "\hat{\partial}"'] \arrow[d, "\lambda"'] \arrow[dr, phantom, "="] & \mathcal{Q}(l) \arrow[d, "\lambda"] \\
\mathcal{P}(l) \arrow[r, "\partial"] & \mathcal{P}(l)
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{Q}(l) \arrow[r, "\oplus_{(\psi^q)_q}(\otimes^{(q,c)\in S}f)\otimes 1", "\widehat{\pi\iota}"'] \arrow[d, "\lambda"'] \arrow[dr, phantom, "="] & \mathcal{Q}(l) \arrow[d, "\lambda"] \\
\mathcal{P}(l) \arrow[r, "\pi\iota"] & \mathcal{P}(l)
\end{tikzcd}
$$

Since $1^{\mathsf{su}}.f = 1^{\mathsf{su}}.g = 1^{\mathsf{su}}$, $1^{\mathsf{su}}.h = 0$, there is a unique map $H : \mathcal{P}(l) \to \mathcal{P}(l)$ of degree -1 such that

$$
\begin{tikzcd}
\mathcal{Q}(l) \arrow[r, "\oplus_{(\psi^q)_q}\sum_{(y,z)\in S}(\otimes^{(q,c)<(y,z)}f)\otimes h\otimes(\otimes^{(q,c)>(y,z)}g)\otimes 1", "\widehat{H}"'] \arrow[d, "\lambda"'] \arrow[dr, phantom, "="] & \mathcal{Q}(l) \arrow[d, "\lambda"] \\
\mathcal{P}(l) \arrow[r, "H"] & \mathcal{P}(l)
\end{tikzcd}
$$

In order to find the commutator $\partial H + H\partial$ we can compute

$$
\begin{aligned}
\widehat{\partial}\widehat{H} + \widehat{H}\widehat{\partial} &= \bigoplus_{(\psi^q:\mathbf{l}^q\hookrightarrow\mathbf{r}^q)_{q=1}^n} \sum_{(y,z)\in S} \Big(\bigotimes_{(q,c)<(y,z)} f\Big) \otimes (f-g) \otimes \Big(\bigotimes_{(q,c)>(y,z)} g\Big) \otimes 1 \\
&= \bigoplus_{(\psi^q:\mathbf{l}^q\hookrightarrow\mathbf{r}^q)_{q=1}^n} \Big(\bigotimes_{(q,c)\in S} f - \bigotimes_{(q,c)\in S} g\Big) \otimes 1 = \widehat{\pi\iota} - 1.
\end{aligned}
$$

Therefore, $\partial H + H\partial = \pi\iota - 1$. □

The projection p' decomposes into a standard trivial cofibration and an epimorphism p''

$$
p' = \big((\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}}) \overset{htis}{\rightarrowtail\!\!\sim\!\!\to} (\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle \mathsf{i}, \mathsf{j}\rangle \xrightarrow{p''} (As1, FAs1_n)\big),
$$

where $p''(1^{\mathsf{su}}) = 1^{\mathsf{su}}$, $p''(\mathsf{i}) = 1^{\mathsf{su}}$, $p''(m_2) = m_2$, $p''(\mathsf{f}_{e_i}) = 1 \in FAs1_n(e_i)$, and other generators go to 0. As a corollary $p''(1^{\mathsf{su}}\rho_\varnothing) = 1^{\mathsf{su}}\rho_\varnothing$. Hence, the projection p'' is a homotopy isomorphism as well.

Generators f_ℓ of the $n\wedge 1$-operad module F_n are interpreted as maps $\mathsf{f}_\ell : \boxtimes^{k\in\mathbf{n}} T^{\ell^k} A_k \to B$ of degree $\deg \mathsf{f}_\ell = 1 - |\ell|$. A cofibrant replacement $(\mathrm{A}_\infty^{hu}, \mathrm{F}_n^{hu}) \to (As1, FAs1_n)$ is constructed as a **gr**-submodule of $(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle \mathsf{i}, \mathsf{j}\rangle$ generated in operadic part by i and g-ary operations of degree $4 - g - 2k$

$$
m_{g_1;g_2;\dots;g_k} = (1^{\otimes g_1} \otimes \mathsf{j} \otimes 1^{\otimes g_2} \otimes \mathsf{j} \otimes \dots \otimes 1^{\otimes g_{k-1}} \otimes \mathsf{j} \otimes 1^{\otimes g_k}) m_{g+k-1},
$$

where $g = \sum_{q=1}^k g_q$, $k \geqslant 1$, $g_q \geqslant 0$, $g + k \geqslant 3$ and in module part by the nullary elements $\mathsf{v}_k = \lambda^k_{e_k}(\mathsf{j}; \mathsf{f}_{e_k}) - \mathsf{j}\rho_\varnothing = \mathsf{j}\mathsf{f}_{e_k} - \mathsf{j}\rho_\varnothing$, $k \in \mathbf{n}$, $\deg \mathsf{v}_k = -1$, and by elements

$$
\begin{aligned}
\mathsf{f}_{(\ell_1^k;\ell_2^k;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}} &= \lambda_{\hat\ell}\big(({}^{\ell_1^k}1, \mathsf{j}, {}^{\ell_2^k}1, \mathsf{j}, \dots, {}^{\ell^k_{t^k-1}}1, \mathsf{j}, {}^{\ell^k_{t^k}}1)_{k\in\mathbf{n}}; \mathsf{f}_{\hat\ell}\big) \\
&= \big[\boxtimes^{k\in\mathbf{n}} T^{\ell^k} A_k \xrightarrow{\boxtimes^{k\in\mathbf{n}}(1^{\otimes\ell_1^k}\otimes\mathsf{j}\otimes 1^{\otimes\ell_2^k}\otimes\mathsf{j}\otimes\dots\otimes 1^{\otimes\ell^k_{t^k-1}}\otimes\mathsf{j}\otimes 1^{\otimes\ell^k_{t^k}})} \boxtimes^{k\in\mathbf{n}} T^{\hat\ell^k} A_k \xrightarrow{\mathsf{f}_{\hat\ell}} B\big]
\end{aligned} \tag{2.31}
$$

of arity $\ell = \big(\sum_{p=1}^{t^k} \ell_p^k\big)_{k\in\mathbf{n}}$, where the intermediate arity is $\hat{\ell} = \big(t_k - 1 + \sum_{p=1}^{t^k} \ell_p^k\big)_{k\in\mathbf{n}} = \big(-1+\sum_{p=1}^{t^k}(\ell_p^k+1)\big)_{k\in\mathbf{n}}$, and of degree $\deg \mathsf{f}_{(\ell_1^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}} = 1+2n-\sum_{k=1}^n\sum_{p=1}^{t^k}(\ell_p^k+2)$. We assume that $t_k \geqslant 1$ for all $k \in \mathbf{n}$ and either $|\hat{\ell}| = \sum_{k=1}^n(t^k-1)+\sum_{k=1}^n\sum_{p=1}^{t^k}\ell_p^k \geqslant 2$, or all $t_k = 1$ and there is $m \in \mathbf{n}$ such that $\ell_1^k = \delta_m^k$. The last condition eliminates from the list the summands $\mathsf{f}_{0,\dots,0,(0;0),0,\dots,0} = \mathsf{j}\mathsf{f}_{e_k}$ of v_k. Setting $\mathsf{i}\partial = 0$, $\mathsf{j}\partial = 1^{\mathsf{su}}-\mathsf{i}$, we turn $(\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i},\mathsf{j}\rangle$ into a **dg**-module and $(\mathrm{A}_\infty^{hu}, \mathrm{F}_n^{hu})$ into its **dg**-submodule. Note that $\mathsf{v}_k\partial = \mathsf{i}\rho_\varnothing - \mathsf{i}\mathsf{f}_{e_k}$.

Let us prove that the graded $n\wedge 1$-module $(\mathrm{A}_\infty^{hu}, \mathrm{F}_n^{hu})$ is free over $(\Bbbk, 0)$. The graded $n\wedge 1$-module $(\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{j}\rangle$ can be presented as

$$
\begin{aligned}
&\big(\mathrm{A}_\infty, \odot_{\geqslant 0}({}^{[n]}\mathrm{A}_\infty; \Bbbk\{\mathsf{f}_\ell \mid \ell\in\mathbb{N}^n - 0\})\big)\langle\mathsf{j}\rangle \\
&\simeq \big(\mathrm{A}_\infty\langle\mathsf{j}\rangle, \odot_{\geqslant 0}({}^{n}\Bbbk\langle m_{n_1;\dots;n_k} \mid k+\textstyle\sum_{q=1}^k n_q \geqslant 3\rangle; \Bbbk\{\mathsf{f}_{(\ell_1^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}} \mid |\hat{\ell}|\geqslant 1\}; \mathrm{A}_\infty\langle\mathsf{j}\rangle)\big). \qquad (2.32)
\end{aligned}
$$

The free graded $n\wedge 1$-operad module generated by $m_{n_1;\dots;n_k}$ and $\mathsf{f}_{(\ell_1^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}}$ has the form

$$
\begin{aligned}
K &= F\big(\Bbbk\{m_{n_1;\dots;n_k} \mid k+\textstyle\sum_{q=1}^k n_q \geqslant 3\}, \Bbbk\{\mathsf{f}_{(\ell_1^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}} \mid |\hat{\ell}|\geqslant 1\}\big) \\
&= \big(\Bbbk\langle m_{n_1;\dots;n_k} \mid k+\textstyle\sum_{q=1}^k n_q \geqslant 3\rangle, \\
&\qquad \odot_{\geqslant 0}\,({}^{[n]}\Bbbk\langle m_{n_1;\dots;n_k} \mid k+\textstyle\sum_{q=1}^k n_q \geqslant 3\rangle; \Bbbk\{\mathsf{f}_{(\ell_1^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}} \mid |\hat{\ell}|\geqslant 1\})\big).
\end{aligned}
$$

It is a direct summand of (2.32), so we have a split exact sequence in $\mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$

$$
0 \longrightarrow K \underset{\pi}{\overset{\alpha}{\rightleftarrows}} (\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{j}\rangle \underset{\omega}{\overset{\varkappa}{\rightleftarrows}} (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) \longrightarrow 0,
$$

where ω takes $\mathsf{j}\rho_\varnothing$ to the nullary generator $\mathsf{j}\rho_\varnothing$. Consider also the graded $n\wedge 1$-module

$$
L = F\big(\Bbbk\{m_{n_1;\dots;n_k} \mid k+\textstyle\sum_{q=1}^k n_q \geqslant 3\}, \Bbbk\{\mathsf{v}_k, \mathsf{f}_{e_k}, \mathsf{f}_{(\ell_1^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}} \mid |\hat{\ell}|\geqslant 2\}\big).
$$

Notice that the map $L \to K$, $\mathsf{v}_k \mapsto \mathsf{j}\mathsf{f}_{e_k}$, which maps other generators identically, identifies the $n\wedge 1$-modules L and K.

Consider the graded module morphism

$$
\begin{aligned}
\beta : L = \big(\Bbbk\langle m_{n_1;\dots;n_k} \mid k+\textstyle\sum_{q=1}^k n_q \geqslant 3\rangle, \overline{L}\big) \to (\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{j}\rangle = \big(\mathrm{A}_\infty\langle\mathsf{j}\rangle, \overline{(\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{j}\rangle}\big), \\
\mathsf{v}_k \mapsto \mathsf{j}\mathsf{f}_{e_k} - \mathsf{j}\rho_\varnothing,
\end{aligned}
$$

which maps other generators identically. The morphism β extends to basic elements so that each factor $\mathsf{j}\rho_\varnothing$ arising from a vertex of type v_k gives its j to subsequent $m_{;;;}$ adding another semicolon to its indexing sequence. This follows by associativity of ρ. The basic elements v_k are mapped by $\beta\varkappa$ to $-\mathsf{j}\rho_\varnothing$. For any other basic element $b(t) \in L$ we have $b(t).\beta\varkappa = 0$.

The map $\beta - \beta\varkappa\omega \in \mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$ factors through α as the following diagram shows:

$$
\begin{tikzcd}
0 \arrow[rr] & & K \arrow[r, shift left, "\alpha"] & (\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{j}\rangle \arrow[l, shift left, "\pi"] \arrow[r, shift left, "\varkappa"] & (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) \arrow[l, shift left, "\omega"] \arrow[r] & 0 \\
& & L \arrow[u, "\exists!\gamma"] \arrow[ur, "\beta-\beta\varkappa\omega"'] & & &
\end{tikzcd}
$$

The unique map $\gamma = (\beta - \beta\varkappa\omega)\pi = \beta\pi : L \to K \in \mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$, such that $\beta - \beta\varkappa\omega = \gamma\alpha$, has a triangular matrix. In fact, L and K have an $\mathbb{N}$-grading, $L^q = \oplus\Bbbk b(t)$, $K^q = \oplus\Bbbk b(t)$, where the summation is over forests t with q vertices labelled by one of v_k (resp. one of jf_{e_k}). The map γ takes the filtration $L_q = L^0 \oplus \cdots \oplus L^q$ to the filtration $K_q = K^0 \oplus \cdots \oplus K^q$. The diagonal entries $\gamma^{qq} : L^q \to K^q$ are identity maps. Thus, the matrix of γ equals $1 - N$, where N is locally nilpotent, and γ is invertible. We obtained a split exact sequence

$$
0 \longrightarrow L \xrightarrow{\beta-\beta\varkappa\omega} (\mathrm{A}_\infty, F_1)\langle\mathsf{j}\rangle \underset{\omega}{\overset{\varkappa}{\rightleftarrows}} (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) \longrightarrow 0. \tag{2.33}
$$

Let us decompose the first two terms into direct sums

$$
\begin{gathered}
\overline{L} = \Bbbk\{\mathsf{v}_s \mid s \in \mathbf{n}\} \oplus \bigl(\overline{L} \ominus \Bbbk\{\mathsf{v}_s \mid s \in \mathbf{n}\}\bigr), \\
\overline{(\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{j}\rangle} = \Bbbk\{\mathsf{j}\rho_\varnothing, \mathsf{jf}_{e_s} \mid s \in \mathbf{n}\} \oplus \bigl(\overline{(\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{j}\rangle} \ominus \Bbbk\{\mathsf{j}\rho_\varnothing, \mathsf{jf}_{e_s} \mid s \in \mathbf{n}\}\bigr),
\end{gathered}
$$

where the complements are spanned by all basic elements except those listed in the first summands. The maps β and $\beta - \beta\varkappa\omega$ preserve this decomposition. Their restriction to the second summand coincide and this is an isomorphism due to exactness of (2.33). If we drop out complements, this split exact sequence takes the form

$$
0 \longrightarrow \Bbbk\{\mathsf{v}_s \mid s \in \mathbf{n}\} \xrightarrow{\beta-\beta\varkappa\omega} \Bbbk\{\mathsf{j}\rho_\varnothing, \mathsf{jf}_{e_s} \mid s \in \mathbf{n}\} \underset{\omega}{\overset{\varkappa}{\rightleftarrows}} \Bbbk\mathsf{j}\rho_\varnothing \longrightarrow 0,
$$

where $\mathsf{v}_s.(\beta - \beta\varkappa\omega) = \mathsf{jf}_{e_s}$. Let us replace it with another split exact sequence

$$
0 \longrightarrow \Bbbk\{\mathsf{v}_s \mid s \in \mathbf{n}\} \underset{\tau}{\overset{\beta}{\rightleftarrows}} \Bbbk\{\mathsf{j}\rho_\varnothing, \mathsf{jf}_{e_s} \mid s \in \mathbf{n}\} \underset{\omega}{\overset{\theta}{\rightleftarrows}} \Bbbk\mathsf{j}\rho_\varnothing \longrightarrow 0,
$$

where $\mathsf{jf}_{e_s}.\theta = \mathsf{j}\rho_\varnothing.\theta = \mathsf{j}\rho_\varnothing$ and $\mathsf{j}\rho_\varnothing.\tau = 0$, $\mathsf{jf}_{e_s}.\tau = \mathsf{v}_s$. Restoring back the dropped isomorphism of second summands we obtain from the above the split exact sequence

$$
0 \longrightarrow L \xrightarrow{\beta} (\mathrm{A}_\infty, F_1)\langle\mathsf{j}\rangle \underset{\omega}{\overset{\theta}{\rightleftarrows}} (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) \longrightarrow 0,
$$

such that θ vanishes on the complement.

Adding freely i we deduce the split exact sequence in $\mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$

$$
0 \to L\langle\mathsf{i}\rangle \to (\mathrm{A}_\infty, \mathrm{F}_n)\langle\mathsf{j}, \mathsf{i}\rangle \to (\Bbbk\mathsf{j}, \Bbbk\mathsf{j}\rho_\varnothing) \to 0. \tag{2.34}
$$

The image of the embedding is precisely $(\mathrm{A}_\infty^{hu},\mathrm{F}_n^{hu})$, thus the latter graded $n\wedge 1$-module is free. In the particular case of $n=0$ the module part is generated by the empty set of generators. Therefore, $\mathrm{F}_0^{hu}=\mathrm{A}_\infty^{hu}(0)$ by Lemma 1.34.

Furthermore, from the top row of diagram (2.30) we deduce a splittable exact sequence in $\mathbf{gr}^{\mathbb{N}\sqcup\mathbb{N}^n}$

$$0\to(\mathrm{A}_\infty,\mathrm{F}_n)\langle\mathsf{i}\rangle\to(\mathrm{A}_\infty^{\mathsf{su}},\mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i}\rangle\to(\Bbbk 1^{\mathsf{su}},\Bbbk 1^{\mathsf{su}}\rho_\varnothing)\to 0.$$

We may choose the splitting of this exact sequence as indicated below:

$$0\to(\mathrm{A}_\infty,\mathrm{F}_n)\langle\mathsf{i}\rangle\to(\mathrm{A}_\infty^{\mathsf{su}},\mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i}\rangle\to(\Bbbk\{1^{\mathsf{su}}-\mathsf{i}\},\Bbbk\{1^{\mathsf{su}}\rho_\varnothing-\mathsf{i}\rho_\varnothing\})\to 0.$$

Adding freely j we get the split exact sequence

$$0\to(\mathrm{A}_\infty,\mathrm{F}_n)\langle\mathsf{i},\mathsf{j}\rangle\to(\mathrm{A}_\infty^{\mathsf{su}},\mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i},\mathsf{j}\rangle\to(\Bbbk\{1^{\mathsf{su}}-\mathsf{i}\},\Bbbk\{(1^{\mathsf{su}}-\mathsf{i})\rho_\varnothing\})\to 0. \tag{2.35}$$

Combining (2.34) with (2.35) we get a split exact sequence

$$0\to(\mathrm{A}_\infty^{hu},\mathrm{F}_n^{hu})\xrightarrow{i'}(\mathrm{A}_\infty^{\mathsf{su}},\mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i},\mathsf{j}\rangle\to(\Bbbk\{1^{\mathsf{su}}-\mathsf{i},\mathsf{j}\},\Bbbk\{(1^{\mathsf{su}}-\mathsf{i})\rho_\varnothing,\mathsf{j}\rho_\varnothing\})\to 0. \tag{2.36}$$

The differential in $(\mathrm{A}_\infty^{hu},\mathrm{F}_n^{hu})$ is computed through that of $(\mathrm{A}_\infty^{\mathsf{su}},\mathrm{F}_n^{\mathsf{su}})\langle\mathsf{i},\mathsf{j}\rangle$. Actually, (2.36) is a split exact sequence in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$, where the third term obtains the differential $\mathsf{j}.\partial=1^{\mathsf{su}}-\mathsf{i}$, $\mathsf{j}\rho_\varnothing.\partial=1^{\mathsf{su}}\rho_\varnothing-\mathsf{i}\rho_\varnothing$. The third term is contractible, which shows that the inclusion i' is a homotopy isomorphism in $\mathbf{dg}^{\mathbb{N}\sqcup\mathbb{N}^n}$. Hence, the epimorphism $p=i'\cdot p''$: $(\mathrm{A}_\infty^{hu},\mathrm{F}_n^{hu})\to(\mathit{As1},\mathit{FAs1}_n)$ is a homotopy isomorphism as well.

In order to prove that $(\mathbb{1},0)\to(\mathrm{A}_\infty^{hu},\mathrm{F}_n^{hu})$ is a standard cofibration we present it as a colimit of sequence of elementary cofibrations

$$(\mathbb{1},0)\to\mathcal{D}_0=F(\Bbbk\{\mathsf{i},m_2\},\Bbbk\{\mathsf{f}_{e_s}\mid s\in\mathbf{n}\})\to\mathcal{D}_1\to\mathcal{D}_2\to\dots,$$

where for $r>0$

$$\mathcal{D}_r=F\bigl(\Bbbk\{\mathsf{i},m_{n_1;\dots;n_k}\mid\deg m_{n_1;\dots;n_k}\geqslant -r\},\Bbbk\{\mathsf{v}_s,\mathsf{f}_{(\ell^k_1;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}}\mid s\in\mathbf{n},\ \deg\mathsf{f}_{(\ell^k_1;\dots;\ell^k_{t^k})}\geqslant -r\}\bigr).$$

Algebra maps over $(\mathrm{A}_\infty^{hu},\mathrm{F}_n^{hu})$ are identified with homotopy unital A_∞-morphisms, which we define in the spirit of Fukaya's approach:

2.20 Definition. A *homotopy unital structure* of an A_∞-morphism $\mathsf{f}:A_1,\dots,A_n\to B$ is an A_∞-morphism $\mathsf{f}^+:(A_k^+)_{k\in\mathbf{n}}=(A_k\oplus\Bbbk 1^{\mathsf{su}}_{A_k}\oplus\Bbbk\mathsf{j}^{A_k})_{k\in\mathbf{n}}\to B\oplus\Bbbk 1^{\mathsf{su}}_B\oplus\Bbbk\mathsf{j}^B=B^+$ between given homotopy unital A_∞-algebra structures such that:

(1) f^+ is a strictly unital: for all $1\leqslant k\leqslant n$

$$1^{\mathsf{su}}_{A_k}\mathsf{f}^+_{e_k}=1^{\mathsf{su}}_B,\quad\bigl[1^{\boxtimes(k-1)}\boxtimes(1^{\otimes a}\otimes 1^{\mathsf{su}}_{A_k}\otimes 1^{\otimes b})\boxtimes 1^{\boxtimes(n-k)}\bigr]\mathsf{f}^+_\ell=0\ \text{ if }\ a+1+b=\ell^k,\ |\ell|>1.$$

(2) the element $\mathsf{v}_k^B=\mathsf{j}^{A_k}\mathsf{f}^+_{e_k}-\mathsf{j}^B$ is contained in B;
(3) the restriction of f^+ to $A_1,\dots,A_n$ gives f;

(4) $\big[\boxtimes^{k\in\mathbf{n}}(A_k\oplus\Bbbk\mathsf{j}^{A_k})^{\otimes\ell^k}\big]\mathsf{f}^+_\ell\subset B$, for each $\ell\in\mathbb{N}^n$, $|\ell|>1$.

Homotopy unital structure of an A_∞-morphism f means a *choice* of such f^+. There is another notion of unitality which is a *property* of an A_∞-morphism:

2.21 Definition (See [BLM08, Proposition 9.13])**.** An A_∞-morphism $\mathsf{f}:A_1,\dots,A_n\to B$ between unital A_∞-algebras is *unital* if the cycles $\mathsf{i}^{A_k}\mathsf{f}_{e_k}$ and i^B differ by a boundary for all $1\leqslant k\leqslant n$.

For a homotopy unital A_∞-morphism $\mathsf{f}:A_1,\dots,A_n\to B$ the equation holds $\mathsf{v}^B_k m_1=\mathsf{v}_k\partial=\mathsf{i}\rho_\varnothing-\mathsf{i}\mathsf{f}_{e_k}=\mathsf{i}^B-\mathsf{i}^{A_k}\mathsf{f}_{e_k}$. Thus an A_∞-morphism with a homotopy unital structure is unital.

2.22 Conjecture. Unitality of an A_∞-morphism is equivalent to homotopy unitality: any unital A_∞-morphism admits a homotopy unital structure.

All reasoning of this section can be applied to F_n in place of F_n. A nullary degree -1 cycle $\mathbf{1}^{\mathsf{su}}$ subject to relations (0.8) is added to A_∞. The resulting operad is denoted A^{su}_∞. We consider the A^{su}_∞-module

$$\tilde{F}_n=\bigodot_{i=1}^n A^{\mathsf{su}}_\infty\odot^i_{A_\infty}F_n\odot^0_{A_\infty}A^{\mathsf{su}}_\infty=\odot_{\geqslant0}({}^nA^{\mathsf{su}}_\infty;\Bbbk\{f_j\mid j\in\mathbb{N}^n-0\};A^{\mathsf{su}}_\infty).$$

It is divided by the graded ideal generated by the following system of relations

$$\rho_\varnothing(\mathbf{1}^{\mathsf{su}})=\lambda^i_{e_i}(\mathbf{1}^{\mathsf{su}};f_{e_i}),\ \forall\,i,\quad \lambda^i_\ell({}^a1,\mathbf{1}^{\mathsf{su}},{}^b1;f_\ell)=0\ \text{ if }\ a+1+b=\ell^i,\ |\ell|>1.$$

The quotient is denoted F^{su}_n. Similarly to the above we add two nullary operations $\mathbf{i}$, $\mathbf{j}$ to A^{su}_∞ with $\deg\mathbf{i}=-1$, $\deg\mathbf{j}=-2$, $\mathbf{i}\partial=0$, $\mathbf{j}\partial=\mathbf{i}-\mathbf{1}^{\mathsf{su}}$. The obtained $A^{\mathsf{su}}_\infty\langle\mathbf{i},\mathbf{j}\rangle$-module $F^{\mathsf{su}}_n\langle\mathbf{i},\mathbf{j}\rangle$ contains an A^{hu}_∞-submodule F^{hu}_n spanned by the nullary elements $\mathbf{v}_k=\lambda^k_{e_k}(\mathbf{j};f_{e_k})-\mathbf{j}\rho_\varnothing=\mathbf{j}f_{e_k}-\mathbf{j}\rho_\varnothing$, $k\in\mathbf{n}$, $\deg\mathbf{v}_k=-2$, and by elements $f_{(\ell^k_1;\ell^k_2;\dots;\ell^k_{t^k})_{k\in\mathbf{n}}}$ similar to (2.31). There are invertible operad module homomorphisms Σ of degree 1 sending $f_j\mapsto\mathsf{f}_j$, $\mathbf{1}^{\mathsf{su}}\mapsto\mathsf{1}^{\mathsf{su}}$, $\mathbf{i}\mapsto\mathsf{i}$, $\mathbf{j}\mapsto\mathsf{j}$, $\mathbf{v}_k\mapsto\mathsf{v}_k$:

$$\Sigma:(A^{\mathsf{su}}_\infty,F^{\mathsf{su}}_n)\to(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_n),\qquad \Sigma:(A^{\mathsf{su}}_\infty,F^{\mathsf{su}}_n)\langle\mathbf{i},\mathbf{j}\rangle\to(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_n)\langle\mathsf{i},\mathsf{j}\rangle,$$
$$\Sigma:(A^{hu}_\infty,F^{hu}_n)\to(\mathrm{A}^{hu}_\infty,\mathrm{F}^{hu}_n).$$

3. Composition of morphisms with several arguments

The mechanism which provides an associative composition of morphisms with several arguments is that of convolution product in the module of linear maps from a coalgebra to an algebra. The part of a coalgebra is played by a colax $\mathcal{C}at$-span multifunctor. A lax $\mathcal{C}at$-span multifunctor $\mathcal{H}om$ comes in place of an algebra. The convolution product of these multifunctors gives a multicategory structure to the collection of A_∞-algebras and A_∞-morphisms with several arguments.

3.1. Colax $\mathcal{C}at$-span multifunctors.

3.2 Definition. A *colax* $\mathcal{C}at$*-span multifunctor* $(F, \psi^I) : (\mathsf{L}, \circledast^I_{\mathsf{L}}, \lambda^f_{\mathsf{L}}, \rho_{\mathsf{L}}) \to (\mathsf{M}, \circledast^I_{\mathsf{M}}, \lambda^f_{\mathsf{M}}, \rho_{\mathsf{M}})$ between lax $\mathcal{C}at$-span multicategories is

i) a 1-morphism $F = ({}_{\mathsf{t}}F, F, {}_{\mathsf{t}}F) : \mathsf{L} = ({}_{\mathsf{t}}\mathsf{L}, \mathsf{L}, {}_{\mathsf{t}}\mathsf{L}) \to ({}_{\mathsf{t}}\mathsf{M}, \mathsf{M}, {}_{\mathsf{t}}\mathsf{M}) = \mathsf{M}$ in $\mathcal{SMQ}_{\mathcal{C}at}$;

ii) a 2-morphism for each set $I \in \operatorname{Ob} \mathcal{O}_{\mathsf{sk}}$

$$
\begin{tikzcd}
\boxdot^I\mathsf{L} \arrow[r, "\boxdot^I F"] \arrow[d, "\circledast^I_{\mathsf{L}}"'] & \boxdot^I\mathsf{M} \arrow[d, "\circledast^I_{\mathsf{M}}"] \\
\mathsf{L} \arrow[r, "F"] \arrow[ur, Rightarrow, "\psi^I"] & \mathsf{M}
\end{tikzcd}
$$

such that

$$
\begin{tikzcd}
\boxdot^{\mathbf{1}}\mathsf{L} \arrow[r, "\boxdot^{\mathbf{1}} F"] \arrow[d, "\circledast^{\mathbf{1}}_{\mathsf{L}}"'] & \boxdot^{\mathbf{1}}\mathsf{M} \arrow[d, "\circledast^{\mathbf{1}}_{\mathsf{M}}"'] \arrow[d, bend left=60, "\mathrm{P}"] \arrow[d, phantom, shift left=3, "\overset{\rho_{\mathsf{M}}}{\Rightarrow}"] \\
\mathsf{L} \arrow[r, "F"] \arrow[ur, Rightarrow, "\psi^{\mathbf{1}}"] & \mathsf{M}
\end{tikzcd}
=
\begin{tikzcd}
\boxdot^{\mathbf{1}}\mathsf{L} \arrow[rr, "\boxdot^{\mathbf{1}} F"] \arrow[d, "\circledast^{\mathbf{1}}_{\mathsf{L}}"'] \arrow[d, bend left=60, "\mathrm{P}"] \arrow[d, phantom, shift left=3, "\overset{\rho_{\mathsf{L}}}{\Rightarrow}"] & & \boxdot^{\mathbf{1}}\mathsf{M} \arrow[d, "\mathrm{P}"] \\
\mathsf{L} \arrow[rr, "F"] \arrow[urr, phantom, "="] & & \mathsf{M}
\end{tikzcd}
$$

and for every map $f : I \to J$ of $\mathcal{O}_{\mathsf{sk}}$ the following equation holds:

$$
\begin{tikzcd}
\boxdot^I\mathsf{L} \arrow[rr, "\boxdot^I F"] \arrow[dr, "\circledast^f_{\mathsf{L}}"] \arrow[dd, "\circledast^I_{\mathsf{L}}"'] & & \boxdot^I\mathsf{M} \arrow[dr, "\circledast^f_{\mathsf{M}}"] & \\
\arrow[r, phantom, "\overset{\lambda^f_{\mathsf{L}}}{\Longrightarrow}"] & \boxdot^J\mathsf{L} \arrow[rr, "\boxdot^J F"] \arrow[dl, "\circledast^J_{\mathsf{L}}"] \arrow[ur, Rightarrow, "\boxdot^{j\in J}\psi^{f^{-1}j}"] & & \boxdot^J\mathsf{M} \arrow[dl, "\circledast^J_{\mathsf{M}}"] \\
\mathsf{L} \arrow[rr, "F"] \arrow[urrr, Rightarrow, "\psi^J"] & & \mathsf{M} &
\end{tikzcd}
=
\begin{tikzcd}
\boxdot^I\mathsf{L} \arrow[r, "\boxdot^I F"] \arrow[dd, "\circledast^I_{\mathsf{L}}"'] & \boxdot^I\mathsf{M} \arrow[dd, "\circledast^I_{\mathsf{M}}"'] \arrow[dr, "\circledast^f_{\mathsf{M}}"] & \\
& \arrow[r, phantom, "\overset{\lambda^f_{\mathsf{M}}}{\Longrightarrow}"] & \boxdot^J\mathsf{M} \arrow[dl, "\circledast^J_{\mathsf{M}}"] \\
\mathsf{L} \arrow[r, "F"] \arrow[uur, Rightarrow, "\psi^I"] & \mathsf{M} &
\end{tikzcd}
$$

Here 2-morphism $\boxdot^{j\in J}\psi^{f^{-1}j}$ means the pasting

$$
\begin{tikzcd}
\boxdot^I\mathsf{L} \arrow[r, "\boxdot^I F"] \arrow[d, "\Lambda^f"'] \arrow[dr, phantom, "="] & \boxdot^I\mathsf{M} \arrow[d, "\Lambda^f"] \\
\boxdot^{j\in J}\boxdot^{f^{-1}j}\mathsf{L} \arrow[r, "\boxdot^{j\in J}\boxdot^{f^{-1}j}F"] \arrow[d, "\boxdot^{j\in J}\circledast^{f^{-1}j}_{\mathsf{L}}"'] & \boxdot^{j\in J}\boxdot^{f^{-1}j}\mathsf{M} \arrow[d, "\boxdot^{j\in J}\circledast^{f^{-1}j}_{\mathsf{M}}"] \\
\boxdot^J\mathsf{L} \arrow[r, "\boxdot^J F"] \arrow[ur, Rightarrow, "\boxdot^{j\in J}\psi^{f^{-1}j}"] & \boxdot^J\mathsf{M}
\end{tikzcd}
$$

We shall show that A_∞-modules F_n form a polymodule cooperad, that is a colax $\mathcal{C}at$-multifunctor $F : \mathsf{F} \to \mathsf{M}$, where the category M of operad polymodules is described in Definition 1.19. Here (strict) $\mathcal{C}at$-operad F has (1-element set of objects), $\mathsf{F}(I) = \mathbb{1}$

is the terminal category for any $I \in \mathcal{O}_{\mathsf{sk}}$, 1-morphism $\circledast^I : \boxdot^I \mathsf{F} \to \mathsf{F}$ is the unique one, 2-morphisms λ^f and ρ are identity morphisms. F is also a $\mathcal{C}at$-span operad.

A general *polymodule cooperad*, that is, a colax $\mathcal{C}at$-multifunctor $F : \mathsf{F} \to \mathsf{M}$ (equivalently, a colax $\mathcal{C}at$-span multifunctor) amounts to the following data: an operad $\mathcal{A} = F(*)$, for each $I \in \mathcal{O}_{\mathsf{sk}}$ an $I \wedge 1$-$\mathcal{A}$-module F_I, for each tree $t : [I] \to \mathcal{O}_{\mathsf{sk}}$ a morphism $\Delta(t) : F_{t(0)} \to \circledast_{\mathsf{M}}(t)(F_{t_h^{-1}b})_{(h,b)\in \mathrm{v}(t)}$ of $t(0) \wedge 1$-$\mathcal{A}$-modules such that for all corollas $t : [1] \to \mathcal{O}_{\mathsf{sk}}$

$$\big(F_{t(0)} \xrightarrow{\Delta(t)} \circledast_{\mathsf{M}}\,(t)(F_{t(0)}) \xrightarrow{\sim} F_{t(0)}\big) = 1, \tag{3.1}$$

for all non-decreasing maps $f : I \to J$, the induced $\psi = [f] : [J] \to [I]$ as in (0.12), and for all trees $t : [I] \to \mathcal{O}_{\mathsf{sk}}$

$$\begin{tikzcd}
F_{t(0)} \arrow[r, "\Delta(t_\psi)"] \arrow[d, "\Delta(t)"'] \arrow[dr, phantom, "="] & \circledast_{\mathsf{M}}(t_\psi)(F_{t_{\psi,g}^{-1}c})_{(g,c)\in \mathrm{v}(t_\psi)} \arrow[d, "\circledast_{\mathsf{M}}(t_\psi)(\Delta(t|^c_{[\psi(g-1),\psi(g)]}))"] \\
\circledast_{\mathsf{M}}(t)(F_{t_h^{-1}b})_{(h,b)\in \mathrm{v}(t)} \arrow[r, "\lambda^f"] & \circledast_{\mathsf{M}}(t_\psi)\big(\circledast_{\mathsf{M}}(t|^c_{[\psi(g-1),\psi(g)]})(F_{t_h^{-1}b})_{(h,b)\in \mathrm{v}(t|^c_{[\psi(g-1),\psi(g)]})}\big)_{(g,c)\in \mathrm{v}(t_\psi)}
\end{tikzcd} \tag{3.2}$$

We are interested in the cases of $\mathcal{A} = A_\infty$, A_∞, A_∞^{hu} and A_∞^{hu}.

3.3 Exercise. Write down explicitly equation (3.2) for the both non-decreasing surjections $\psi : [2] \to [1]$ and a tree $t : [1] \to \mathcal{O}_{\mathsf{sk}}$. Conclude that for the tree $r : [0] \to \mathcal{O}_{\mathsf{sk}}$ the operad $\mathcal{A}$-bimodule map $\Delta(r) : F_1 \to \mathcal{A}$ plays the part of a counit for Δ.

Viewing (system of $\mathcal{A}$-modules) $F : \mathsf{F} \to \mathsf{M}$ as a coalgebra and $\mathcal{H}om : \mathsf{B} \to \mathsf{M}$ (coming from a symmetric multicategory C) as an algebra we consider homomorphisms between them (in the sense of M) and they have to form an algebra as well. So we define a multiquiver H whose objects are $\mathcal{A}$-algebras $(B, \alpha_B : \mathcal{A} \to \mathcal{E}nd\, B)$ with

$$\begin{aligned}
&\mathsf{H}((A_i, \alpha_{A_i})_{i\in I}; (B, \alpha_B)) \\
&\qquad = \{((\alpha_{A_i})_{i\in I}; \phi; \alpha_B) \in \mathsf{M}\big(({}^I\mathcal{A}; F_I; \mathcal{A}), ((\mathcal{E}nd\, A_i)_{i\in I}; hom((A_i)_{i\in I}; B); \mathcal{E}nd\, B)\big)\}.
\end{aligned}$$

Let us define a multicategory composition for it. For any tree t and any collection of $\mathcal{A}$-algebras $\alpha_h^b : \mathcal{A} \to \mathcal{E}nd\, A_h^b$, $(h,b) \in \overline{\mathrm{v}}(t)$, assume given $t_h^{-1}b \wedge 1$-operad module morphisms for $(h,b) \in \mathrm{v}(t)$:

$$\begin{aligned}
&\big((\alpha_{h-1}^a)_{a\in t_h^{-1}b}; g_h^b; \alpha_h^b\big) : \big({}^{t_h^{-1}b}\mathcal{A}; F_{t_h^{-1}b}; \mathcal{A}\big) \\
&\qquad\qquad \to \big((\mathcal{E}nd\, A_{h-1}^a)_{a\in t_h^{-1}b}; hom((A_{h-1}^a)_{a\in t_h^{-1}b}; A_h^b); \mathcal{E}nd\, A_h^b\big).
\end{aligned}$$

Then their composition is defined as $\big((\alpha_0^a)_{a\in t(0)}; \mathrm{comp}(t)(g_h^b); \alpha_{\max[I]}^1\big)$, where

$$\begin{aligned}
\mathrm{comp}(t)(g_h^b) = \big[F_{t(0)} &\xrightarrow{\Delta(t)} \circledast_{\mathsf{M}}\,(t)(F_{t_h^{-1}b})_{(h,b)\in \mathrm{v}(t)} \xrightarrow{\circledast_{\mathsf{M}}(t)(g_h^b)} \\
&\circledast_{\mathsf{M}}\,(t)\big(\mathcal{H}om((A_{h-1}^a)_{a\in t_h^{-1}b}; A_h^b)\big)_{(h,b)\in \mathrm{v}(t)} \xrightarrow{\mathrm{comp}(t)} \mathcal{H}om((A_0^a)_{a\in t(0)}; A_{\max[I]}^1)\big].
\end{aligned} \tag{3.3}$$

3.4 Proposition. *The composition in H is strictly associative.*

Proof. For any $f : I \to J$ and $(y,c) \in \mathrm{v}(t_\psi)$ in notation from Section 0.27.1 denote

$$h_y^c = \big[F_{t_{\psi,g}^{-1}c} \xrightarrow{\Delta(t^{|c}_{[\psi(y-1),\psi y]})} \circledast_{\mathsf{M}} (t^{|c}_{[\psi(y-1),\psi y]})(F_{t_x^{-1}b})_{(x,b)\in \mathrm{v}(t^{|c}_{[\psi(y-1),\psi y]})}$$
$$\xrightarrow{\circledast_{\mathsf{M}}(t^{|c}_{[\psi(y-1),\psi y]})(g_x^b)} \circledast_{\mathsf{M}} (t^{|c}_{[\psi(y-1),\psi y]})\big(\mathcal{H}om((A^a_{x-1})_{a\in t_x^{-1}b}; A^b_x)\big)_{(x,b)\in \mathrm{v}(t^{|c}_{[\psi(y-1),\psi y]})}$$
$$\xrightarrow{\mathrm{comp}(t^{|c}_{[\psi(y-1),\psi y]})} \mathcal{H}om((A^a_{\psi(y-1)})_{a\in t_{\psi,g}^{-1}c}; A^c_{\psi y})\big].$$

We plug in this expression into

$$\big[F_{t(0)} \xrightarrow{\Delta(t_\psi)} \circledast_{\mathsf{M}} (t_\psi)(F_{t_{\psi,g}^{-1}c})_{(y,c)\in \mathrm{v}(t_\psi)} \xrightarrow{\circledast_{\mathsf{M}}(t_\psi)(h_y^c)}$$
$$\circledast_{\mathsf{M}} (t_\psi)\big(\mathcal{H}om((A^a_{\psi(y-1)})_{a\in t_{\psi,g}^{-1}c}; A^c_{\psi y})\big)_{(y,c)\in \mathrm{v}(t_\psi)} \xrightarrow{\mathrm{comp}(t_\psi)} \mathcal{H}om((A^a_0)_{a\in t(0)}; A^1_{\max[I]})\big]$$
$$= \big[F_{t(0)} \xrightarrow{\Delta(t_\psi)} \circledast_{\mathsf{M}} (t_\psi)(F_{t_{\psi,g}^{-1}c})_{(y,c)\in \mathrm{v}(t_\psi)} \xrightarrow{\circledast_{\mathsf{M}}(t_\psi)(\Delta(t^{|c}_{[\psi(y-1),\psi y]}))}$$
$$\circledast_{\mathsf{M}} (t_\psi)\big(\circledast_{\mathsf{M}}(t^{|c}_{[\psi(y-1),\psi y]})(F_{t_x^{-1}b})_{(x,b)\in \mathrm{v}(t^{|c}_{[\psi(y-1),\psi y]})}\big)_{(y,c)\in \mathrm{v}(t_\psi)} \xrightarrow{\circledast_{\mathsf{M}}(t_\psi)\circledast_{\mathsf{M}}(t^{|c}_{[\psi(y-1),\psi y]})(g_x^b)}$$
$$\circledast_{\mathsf{M}} (t_\psi)\big(\circledast_{\mathsf{M}}(t^{|c}_{[\psi(y-1),\psi y]})\big(\mathcal{H}om((A^a_{x-1})_{a\in t_x^{-1}b}; A^b_x)\big)_{(x,b)\in \mathrm{v}(t^{|c}_{[\psi(y-1),\psi y]})}\big)_{(y,c)\in \mathrm{v}(t_\psi)}$$
$$\xrightarrow{\circledast_{\mathsf{M}}(t_\psi)\,\mathrm{comp}(t^{|c}_{[\psi(y-1),\psi y]})} \circledast_{\mathsf{M}} (t_\psi)\big(\mathcal{H}om((A^a_{\psi(y-1)})_{a\in t_{\psi,g}^{-1}c}; A^c_{\psi y})\big)_{(y,c)\in \mathrm{v}(t_\psi)}$$
$$\xrightarrow{\mathrm{comp}(t_\psi)} \mathcal{H}om((A^a_0)_{a\in t(0)}; A^1_{\max[I]})\big]$$
$$= \big[F_{t(0)} \xrightarrow{\Delta(t)} \circledast_{\mathsf{M}} (t)(F_{t_x^{-1}b})_{(x,b)\in \mathrm{v}(t)} \xrightarrow{\lambda^f_{\mathsf{M}}}$$
$$\circledast_{\mathsf{M}} (t_\psi)\big(\circledast_{\mathsf{M}}(t^{|c}_{[\psi(y-1),\psi y]})(F_{t_x^{-1}b})_{(x,b)\in \mathrm{v}(t^{|c}_{[\psi(y-1),\psi y]})}\big)_{(y,c)\in \mathrm{v}(t_\psi)} \xrightarrow{\circledast_{\mathsf{M}}(t_\psi)\circledast_{\mathsf{M}}(t^{|c}_{[\psi(y-1),\psi y]})(g_x^b)}$$
$$\circledast_{\mathsf{M}} (t_\psi)\big(\circledast_{\mathsf{M}}(t^{|c}_{[\psi(y-1),\psi y]})\big(\mathcal{H}om((A^a_{x-1})_{a\in t_x^{-1}b}; A^b_x)\big)_{(x,b)\in \mathrm{v}(t^{|c}_{[\psi(y-1),\psi y]})}\big)_{(y,c)\in \mathrm{v}(t_\psi)}$$
$$\xrightarrow{\circledast_{\mathsf{M}}(t_\psi)\,\mathrm{comp}(t^{|c}_{[\psi(y-1),\psi y]})} \circledast_{\mathsf{M}} (t_\psi)\big(\mathcal{H}om((A^a_{\psi(y-1)})_{a\in t_{\psi,g}^{-1}c}; A^c_{\psi y})\big)_{(y,c)\in \mathrm{v}(t_\psi)}$$
$$\xrightarrow{\mathrm{comp}(t_\psi)} \mathcal{H}om((A^a_0)_{a\in t(0)}; A^1_{\max[I]})\big]$$
$$= \big[F_{t(0)} \xrightarrow{\Delta(t)} \circledast_{\mathsf{M}} (t)(F_{t_x^{-1}b})_{(x,b)\in \mathrm{v}(t)} \xrightarrow{\circledast_{\mathsf{M}}(t)(g_x^b)} \circledast_{\mathsf{M}} (t)\big(\mathcal{H}om((A^a_{x-1})_{a\in t_x^{-1}b}; A^b_x)\big)_{(x,b)\in \mathrm{v}(t)}$$
$$\xrightarrow{\lambda^f_{\mathsf{M}}} \circledast_{\mathsf{M}} (t_\psi)\big(\circledast_{\mathsf{M}}(t^{|c}_{[\psi(y-1),\psi y]})\big(\mathcal{H}om((A^a_{x-1})_{a\in t_x^{-1}b}; A^b_x)\big)_{(x,b)\in \mathrm{v}(t^{|c}_{[\psi(y-1),\psi y]})}\big)_{(y,c)\in \mathrm{v}(t_\psi)}$$
$$\xrightarrow{\circledast_{\mathsf{M}}(t_\psi)\,\mathrm{comp}(t^{|c}_{[\psi(y-1),\psi y]})} \circledast_{\mathsf{M}} (t_\psi)\big(\mathcal{H}om((A^a_{\psi(y-1)})_{a\in t_{\psi,g}^{-1}c}; A^c_{\psi y})\big)_{(y,c)\in \mathrm{v}(t_\psi)}$$
$$\xrightarrow{\mathrm{comp}(t_\psi)} \mathcal{H}om((A^a_0)_{a\in t(0)}; A^1_{\max[I]})\big]$$

$$= \left[F_{t(0)} \xrightarrow{\Delta(t)} \circledast_{\mathsf{M}} (t)(F_{t_x^{-1}b})_{(x,b)\in\mathrm{v}(t)} \xrightarrow{\circledast_{\mathsf{M}}(t)(g_x^b)} \circledast_{\mathsf{M}} (t)\big(\mathcal{H}om((A_{x-1}^a)_{a\in t_x^{-1}b}; A_x^b)\big)_{(x,b)\in\mathrm{v}(t)}\right.$$
$$\left.\xrightarrow{\mathrm{comp}(t)} \mathcal{H}om((A_0^a)_{a\in t(0)}; A^1_{\max[I]})\right] = \mathrm{comp}(t)(g_h^b).$$

Here we have used (3.2), naturality of λ_{M}, equation (1.27) and definition (3.3). □

Below we shall prove that the convolution H of A_∞-module cooperad F and the lax $\mathcal{C}at$-multifunctor $\mathcal{H}om$ built from $\underline{\mathsf{C}}_{\Bbbk}$ gives a multicategory of A_∞-algebras and A_∞-morphisms. Its objects are A_∞-algebras and morphisms $(A_i)_{i\in I} \to B \in \mathsf{H}$ are morphisms of $n \wedge 1$-operad modules

$$({}^I\mathrm{A}_\infty; \mathrm{F}_n; \mathrm{A}_\infty) \to ((\mathcal{E}nd\, A_i)_{i\in I}; \mathcal{H}om((A_i)_{i\in I}; B); \mathcal{E}nd\, B),$$

which are precisely A_∞-morphisms with several arguments. Their composition is the composition in H.

3.5. Comultiplication under A_∞. Taking tensor coalgebra of a graded $\Bbbk$-module gives morphism of multiquivers to multicategory of differential graded augmented counital coassociative coalgebras $Ts : \mathsf{H} \hookrightarrow \mathbf{dgac}$. We wish to define a colax $\mathcal{C}at$-span multifunctor $F : \mathsf{F} \to \mathsf{M}$ such that Ts becomes a multifunctor. The following statements follow from results of [BLM08].

3.6 Proposition (See Proposition 6.8 of [BLM08])**.** *A $T^{\geqslant 1}$-coalgebra C in* **dg** *is a coassociative coalgebra $(C, \overline{\Delta} : C \to C \otimes C)$ in* **dg** *such that*

$$C = \operatorname*{colim}_{k\to\infty} \mathrm{Ker}\big(\overline{\Delta}^{(k)} : C \to C^{\otimes k}\big).$$

3.7 Corollary (See Corollary 6.11 of [BLM08])**.** *Let C be a $T^{\geqslant 1}$-coalgebra in* **dg**, *and let $B \in \mathrm{Ob}\,\mathbf{dg}$ be a complex. Then there is a natural bijection*

$$\mathbf{dg}_{T^{\geqslant 1}}(C, T^{\geqslant 1}B) \to \mathbf{dg}(C, B), \qquad (f : C \to T^{\geqslant 1}B) \mapsto \big(C \xrightarrow{f} T^{\geqslant 1}B \xrightarrow{\mathrm{pr}_1} B\big),$$

where $\mathbf{dg}_{T^{\geqslant 1}}$ is the category of $T^{\geqslant 1}$-coalgebras in **dg**.

3.8 Proposition (See Corollary 6.18 and Proposition 6.19 of [BLM08])**.** *The full and faithful functor*

$$T^{\leqslant 1} : \mathbf{dg}_{T^{\geqslant 1}} \to \mathbf{dgac}, \qquad (C, \overline{\Delta}) \mapsto$$
$$\big(\Bbbk \oplus C, \Delta_0 = \mathrm{pr}_1 \cdot \overline{\Delta} \cdot (\mathrm{in}_1 \otimes \mathrm{in}_1) + \mathrm{id} \otimes \mathrm{in}_0 + \mathrm{in}_0 \otimes \mathrm{id} - \mathrm{pr}_0 \cdot (\mathrm{in}_0 \otimes \mathrm{in}_0), \varepsilon = \mathrm{pr}_0, \eta = \mathrm{in}_0\big)$$

makes $\mathbf{dg}_{T^{\geqslant 1}}$ into a symmetric Monoidal subcategory of **dgac**.

3.9 Corollary. *An arbitrary augmented* **dg**-*coalgebra $A = \otimes^{i\in I} TA_i$ comes from a $T^{\geqslant 1}$-coalgebra $A \ominus \Bbbk$ and there is a natural bijection*

$$\mathbf{dgac}(\otimes^{i\in I} TA_i, TB) \to \mathbf{dg}((\otimes^{i\in I} TA_i) \ominus \Bbbk, B),$$
$$(f : \otimes^{i\in I} TA_i \to TB) \mapsto \big((\otimes^{i\in I} TA_i) \ominus \Bbbk \xrightarrow{f|} T^{\geqslant 1}B \xrightarrow{\mathrm{pr}_1} B\big).$$

Let us denote by a_∞ the multiquiver of A_∞-algebras and their morphisms. It admits a full and faithful embedding $Ts : \mathsf{a}_\infty \hookrightarrow \mathbf{dgac}$ into the multiquiver of differential graded augmented counital coassociative coalgebras over $\Bbbk$. Actually the latter is a multicategory and a_∞ is isomorphic to its submulticategory. In this way F obtains a unique colax $\mathcal{C}at$-span multifunctor structure $\Delta(t)$. Let us describe the details.

Objects of $\mathsf{a}_\infty = \mathsf{H}$, that is, A_∞-algebras $(B, \alpha_B : \mathrm{A}_\infty \to \mathcal{E}nd\, B)$ are taken by Ts to the tensor coalgebra $(TsB, \Delta_0, \varepsilon, \eta)$, where $TsB = \oplus_{n=0}^\infty T^n sB = \oplus_{n=0}^\infty (B[1])^{\otimes n}$, Δ_0 is the cut comultiplication

$$\Delta_0(x_1 \otimes \cdots \otimes x_n) = \sum_{i=0}^n (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n),$$

$\varepsilon = \mathrm{pr}_0 : TsB \longrightarrow T^0 sB = \Bbbk$ is the counit and $\eta = \mathrm{in}_0 : \Bbbk = T^0 sB \hookrightarrow TsB$ is the augmentation.

The differential $b : TsB \to TsB$, $\deg b = 1$, has matrix entries $b^{n,k} : T^n sB \to T^k sB$,

$$b^{n,k} = \sum_{a+p+c=n}^{a+1+c=k} 1^{\otimes a} \otimes b_p \otimes 1^{\otimes c}, \quad \text{where} \quad b_p = (-1)^n (\sigma^{\otimes p})^{-1} \cdot m_p \cdot \sigma : T^p sB \to sB$$

for $p \geqslant 1$, $b_0 = 0$, and $s : B \to sB = B[1]$, $x \mapsto x$, is the shift map (the suspension), $\deg s = -1$, $\deg b_p = 1$. Here $m_p : T^p B \to B$, $\deg m_p = 2-p$, are linear maps representing generators $m_p \in \mathrm{A}_\infty(p)$ for $p \geqslant 2$ and $m_1 : B \to B$ is the differential in the complex B.

Morphisms of $\mathsf{a}_\infty = \mathsf{H}$, A_∞-algebra morphisms $\mathsf{f} : (A_i)_{i\in\mathbf{n}} \to B$, are taken by Ts to the augmented coalgebra chain homomorphisms $f : \otimes^{i\in\mathbf{n}} TsA_i \to TsB$, whose compositions with the projections $\mathrm{pr}_l : TsB \to T^l sB$ are given by

$$\begin{aligned} f \cdot \mathrm{pr}_l = \big[TsA_1 \otimes \cdots \otimes TsA_n &\xrightarrow{\Delta_0^{(l)} \otimes \cdots \otimes \Delta_0^{(l)}} (TsA_1)^{\otimes l} \otimes \cdots \otimes (TsA_n)^{\otimes l} \\ &\xrightarrow{\overline{\varkappa}_{n,l}} (TsA_1 \otimes \cdots \otimes TsA_n)^{\otimes l} \xrightarrow{\check{f}^{\otimes l}} (sB)^{\otimes l} \big], \end{aligned} \tag{3.4}$$

where the restriction of $\check{f}$ to $T^{j^1} sA_1 \otimes \cdots \otimes T^{j^n} sA_n$ is given by the component

$$f_j = (\sigma^{\otimes j^1} \otimes \cdots \otimes \sigma^{\otimes j^n})^{-1} \cdot \mathsf{f}_j \cdot \sigma : T^{j^1} sA_1 \otimes \cdots \otimes T^{j^n} sA_n \to sB, \tag{3.5}$$

f_j being linear maps that represent the generators $\mathsf{f}_j \in \mathrm{F}_n(j^1, \dots, j^n)$. The symmetry $\overline{\varkappa}_{n,l} = c_{s_{n,l}}$ corresponds to the permutation $s_{n,l}$ of the set $\{1, 2, \dots, nl\}$,

$$s_{n,l}(1 + t + kl) = 1 + k + tn \quad \text{for} \quad 0 \leqslant t < l,\ 0 \leqslant k < n.$$

Detailed description of map (3.4) on direct summands is

$$\begin{aligned} \Big[\bigotimes_{a\in\mathbf{n}} T^{j^a} sA_a &\xrightarrow{\otimes^{\mathbf{n}} \Delta_0^{(l)}} \bigotimes_{a\in\mathbf{n}} \bigoplus_{\sum_{q=1}^l r_q^a = j^a} \bigotimes_{p\in\mathbf{l}} T^{r_p^a} sA_a \xrightarrow{\sim} \bigoplus_{\sum_{q=1}^l r_q^a = j^a} \bigotimes_{a\in\mathbf{n}} \bigotimes_{p\in\mathbf{l}} T^{r_p^a} sA_a \\ &\xrightarrow{\oplus \overline{\varkappa}_{n,l}} \bigoplus_{\sum_{q=1}^l r_q^a = j^a} \bigotimes_{p\in\mathbf{l}} \bigotimes_{a\in\mathbf{n}} T^{r_p^a} sA_a \xrightarrow{\sum \otimes^{p\in\mathbf{l}} f_{(r_p^a)_{a\in\mathbf{n}}}} \bigotimes_{p\in\mathbf{l}} sB = T^l sB \Big]. \end{aligned} \tag{3.6}$$

Due to reasoning after equation (8.20.2) in [BLM08] the map

$$Ts : \mathsf{a}_\infty((A_i)_{i\in\mathbf{n}};B) \to \mathbf{dgac}(\otimes^{i\in\mathbf{n}}TsA_i, TsB), \quad \mathsf{f} \mapsto f,$$

is a bijection. Thus, for an arbitrary tree $t : [I] \to \mathcal{O}_{\mathsf{sk}}$ and arbitrary vertex $(h,b) \in \mathrm{v}(t)$

$$Ts : \mathsf{a}_\infty((A^a_{h-1})_{a\in t_h^{-1}b}; A^b_h) \to \mathbf{dgac}(\otimes^{a\in t_h^{-1}b}TsA^a_{h-1}, TsA^b_h), \quad g^b_h = (g^b_{h,j})_{j\in\mathbb{N}^n} \mapsto \hat{g}^b_h,$$

is a bijection. Define composition in a_∞ as

$$\Big[\prod_{(h,b)\in\mathrm{v}(t)} \mathsf{a}_\infty((A^a_{h-1})_{a\in t_h^{-1}b}; A^b_h) \xrightarrow[\sim]{\prod Ts} \prod_{(h,b)\in\mathrm{v}(t)} \mathbf{dgac}(\otimes^{a\in t_h^{-1}b}TsA^a_{h-1}, TsA^b_h)$$
$$\xrightarrow{\mathrm{comp}_{\mathbf{dgac}}(t)} \mathbf{dgac}(\otimes^{a\in t(0)}TsA^a_0, TsA^1_{\max[I]}) \xrightarrow[\sim]{} \mathsf{a}_\infty((A^a_0)_{a\in t(0)}; A^1_{\max[I]})\Big],$$
$$(g^b_h)_{(h,b)\in\mathrm{v}(t)} \mapsto \mathrm{comp}(t)(g^b_h), \quad \text{where}$$

$$\mathrm{comp}(t)(g^b_h)_j = \Big[\otimes^{a\in t(0)}T^{j^a}A^a_0 \xrightarrow{\otimes^{a\in t(0)}\sigma^{\otimes j^a}} \otimes^{a\in t(0)} T^{j^a}sA^a_0 \xrightarrow{\otimes^{b_1\in t(1)}\widehat{g^{b_1}_1}} \otimes^{b_1\in t(1)} T^{j^{b_1}_1}sA^{b_1}_1$$
$$\xrightarrow{\otimes^{b_2\in t(2)}\widehat{g^{b_2}_2}} \otimes^{b_2\in t(2)} T^{j^{b_2}_2}sA^{b_2}_2 \to \dots \xrightarrow{g^1_{\max[I]}} sA^1_{\max[I]} \xrightarrow{\sigma^{-1}} A^1_{\max[I]}\Big], \quad (3.7)$$

and g^b_h is given via its components

$$\mathsf{g}^b_{h,j} : \mathrm{F}_{t_h^{-1}b}(j) \to \underline{\mathbf{dg}}(\otimes^{a\in t_h^{-1}b}T^{j^a}A^a_{h-1}, A^b_h), \quad \mathsf{f}_j \mapsto \big(\mathsf{g}^b_{h,j} : \otimes^{a\in t_h^{-1}b}T^{j^a}A^a_{h-1} \to A^b_h\big).$$

Here j belongs to $\mathbb{N}^{t_h^{-1}b}$.

We are going to show that expression (3.7) is the image of f_j under the left-bottom path

$$\big[F_{t(0)}(j) \xrightarrow{\Sigma(j)} \mathrm{F}_{t(0)}(j) \xrightarrow{\Delta^{\mathsf{G}}(t)} \circledast_{\mathsf{G}}(t)(\mathrm{F}_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}(j) \xrightarrow{\circledast_{\mathsf{G}}(t)(\mathsf{g}^b_h)}$$
$$\circledast_{\mathsf{G}}(t)(hom((A^a_{h-1})_{a\in t_h^{-1}b}; A^b_h))_{(h,b)\in\mathrm{v}(t)}(j) \xrightarrow{\mathrm{comp}(t)} hom((A^a_0)_{a\in t(0)}; A^1_{\max[I]})(j)\big]$$

of (3.8). This diagram uses the invertible homomorphism $\Sigma : F_n \to \mathrm{F}_n$ of degree 1 defined by (2.15).

Let $j \in \mathbb{N}^{t(0)}$ and let τ denote a t-tree $t \to \mathcal{O}_{\mathsf{sk}}$ such that $|\tau(0,a)| = j^a$ for all $a \in t(0)$. Shorten $\tau_{(h-1,a)\to(h,b)} : \tau(h-1,a) \to \tau(h,b)$ to $\tau_{(h-1,a)}$. By definition

$$\circledast_{\mathsf{G}}(t)(F_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}(j) = \bigoplus_{\forall a\in t(0)\,|\tau(0,a)|=j^a}^{t-\text{tree }\tau} \bigotimes^{h\in I} \bigotimes^{b\in t(h)} \bigotimes^{p\in\tau(h,b)} F_{t_h^{-1}b}\Big(|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}\Big), \quad (3.9)$$

$$\circledast_{\mathsf{G}}(t)(hom((A^a_{h-1})_{a\in t_h^{-1}b}; A^b_h))_{(h,b)\in\mathrm{v}(t)}(j)$$
$$= \bigoplus_{\forall a\in t(0)\,|\tau(0,a)|=j^a}^{t-\text{tree }\tau} \bigotimes^{h\in I} \bigotimes^{b\in t(h)} \bigotimes^{p\in\tau(h,b)} \underline{\mathbf{dg}}(\otimes^{a\in t_h^{-1}b}T^{|\tau^{-1}_{(h-1,a)}(p)|}A^a_{h-1}, A^b_h).$$

$$
\begin{tikzcd}
F_{t(0)}(j) \arrow[r, "\Delta(t)"] \arrow[d, "\Sigma(j)"'] \arrow[dr, phantom, "((-1)^{c(\tilde{\tau})})_{\tau}"] & \circledast_{\mathsf{G}}(t)(F_{t_h^{-1}b})_{(h,b)\in \mathrm{v}(t)}(j) \arrow[r, "\circledast_{\mathsf{G}}(t)(g_h^b)"] \arrow[d, "\circledast_{\mathsf{G}}(t)(\Sigma)(j)"] \arrow[dr, phantom, "="] & \circledast_{\mathsf{G}}(t)(hom((sA^a_{h-1})_{a\in t_h^{-1}b}; sA^b_h))_{(h,b)\in \mathrm{v}(t)}(j) \arrow[r, "\mathrm{comp}(t)"] \arrow[d, "\circledast_{\mathsf{G}}(t)(hom((\sigma)_{a\in t_h^{-1}b}; \sigma^{-1}))_{(h,b)\in \mathrm{v}(t)}(j)" description] \arrow[dr, phantom, "((-1)^{c(\tilde{\tau})})_{\tau}"] & hom((sA^a_0)_{a\in t(0)}; sA^1_{\max[I]})(j) \arrow[d, "hom((\sigma)_{a\in t(0)}; \sigma^{-1})(j)" description] \\
\mathrm{F}_{t(0)}(j) \arrow[r, "\Delta(t)"] & \circledast_{\mathsf{G}}(t)(\mathrm{F}_{t_h^{-1}b})_{(h,b)\in \mathrm{v}(t)}(j) \arrow[r, "\circledast_{\mathsf{G}}(t)(\mathsf{g}_h^b)"] & \circledast_{\mathsf{G}}(t)(hom((A^a_{h-1})_{a\in t_h^{-1}b}; A^b_h))_{(h,b)\in \mathrm{v}(t)}(j) \arrow[r, "\mathrm{comp}(t)"] & hom((A^a_0)_{a\in t(0)}; A^1_{\max[I]})(j)
\end{tikzcd}
\tag{3.8}
$$

Diagram (2.16) implies commutativity of

$$\begin{tikzcd}
F_{t_h^{-1}b} \arrow[r, "g_h^b"] \arrow[d, "\Sigma"'] & hom((sA_{h-1}^a)_{a\in t_h^{-1}b}; sA_h^b) \arrow[d, "hom((\sigma)_{a\in t_h^{-1}b};\sigma^{-1})"] \\
\mathrm{F}_{t_h^{-1}b} \arrow[r, "\mathsf{g}_h^b"] & hom((A_{h-1}^a)_{a\in t_h^{-1}b}; A_h^b)
\end{tikzcd}$$

Here $hom((\sigma)_{a\in t_h^{-1}b};\sigma^{-1}) = hom((\sigma)_{a\in t_h^{-1}b};1)\cdot hom((1)_{a\in t_h^{-1}b};\sigma^{-1})$ is the product of right operators. Hence, the middle square of (3.8) is commutative.

The second and the third term of both rows of diagram (3.8) are direct sums over τ. The diagram splits into a direct sum over τ of 3-squares-diagrams in which the second square is commutative, while the third square commutes up to the sign $(-1)^{c(\tilde{\tau})}$. The mapping $\Delta^{\mathsf{G}}(t)$ is defined so that the first square of the diagram commutes up to the same sign $(-1)^{c(\tilde{\tau})}$. Thus, the exterior of diagram (3.8) is commutative.

3.10 Proposition. *Define for a tree $t:[I]\to\mathcal{O}_{\mathsf{sk}}$ the degree 0 graded $t(0)\wedge 1$-A_∞-module homomorphism $\Delta^{\mathsf{G}}(t)(j): F_{t(0)}(j)\to \circledast_{\mathsf{G}}(t)(F_{t_h^{-1}b})_{(h,b)\in \mathrm{v}(t)}(j)$ on generators as*

$$\Delta^{\mathsf{G}}(t)(f_j) = \sum_{\forall a\in t(0)\; |\tau(0,a)|=j^a}^{t-tree\;\tau} \otimes^{h\in I}\otimes^{b\in t(h)}\otimes^{p\in\tau(h,b)} f_{|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}}. \tag{3.10}$$

In particular, for the only tree t with empty I the A_∞-bimodule homomorphism $\Delta^{\mathsf{G}}(t)(j)$: $F_1(j)\to\Bbbk(j)=\delta_{j1}\Bbbk$ is determined by $\Delta^{\mathsf{G}}(t)(f_j)=\delta_{j1}$, $j\geqslant 1$. Then this comultiplication is coassociative and expression (3.7) *is the image of f_j under the left-bottom path of* (3.8)*. Thus, the canonical actions of the unit operad $\Bbbk$ on $F_\bullet$ make $(\Bbbk, F_\bullet, \Delta^{\mathsf{G}})$ into a graded polymodule cooperad.*

A tree $r:[I]\to\mathcal{O}_{\mathsf{sk}}$ is *surjective* if mappings $r_h: r(h-1)\to r(h)$ are surjective for all $h\in I$. The summation in expression (3.9) and formula (3.10) extends precisely over t-trees τ such that all trees $\tilde{\tau}$ are surjective. In fact, $F_\varnothing = 0$ and the summand of (3.9) corresponding to τ does not vanish iff

$$\forall\, h\in I\;\forall\, b\in t(h)\;\forall\, p\in\tau(h,b)\;\exists\, a\in t_h^{-1}b\quad \tau^{-1}_{(h-1,a)}(p)\neq\varnothing.$$

Equivalently, for all $h\in I$

$$\forall\,(b,p)\in\bigsqcup_{b\in t(h)}\tau(h,b)=\tilde{\tau}(h)\;\exists\,(a,q)\in\bigsqcup_{a\in t(h-1)}\tau(h-1,a)=\tilde{\tau}(h-1)\;\;(t_h a,\tau_{(h-1,a)}q)=(b,p).$$

The last equation means that $\tilde{\tau}_h(a,q)=(b,p)$, that is, $\tilde{\tau}$ is surjective.

The number of surjective trees $\tilde{\tau}$ is finite. Hence, the number of tree mappings $\tilde{\tau}\to t$ is finite, and the number of t-trees τ with this surjectivity property is finite as well. Thus the sum is finite.

Deduced comultiplication $\Delta^{\mathsf{G}}(t)(j) : \mathrm{F}_{t(0)}(j) \to \circledast_{\mathsf{G}}(t)(\mathrm{F}_{t_h^{-1}b})_{(h,b)\in\mathsf{v}(t)}(j)$ has degree 0:

$$\Delta^{\mathsf{G}}(t)(\mathsf{f}_j) = \sum_{\forall a\in t(0)\,|\tau(0,a)|=j^a}^{t-\text{tree }\tau} (-1)^{c(\tilde{\tau})} \otimes^{h\in I} \otimes^{b\in t(h)} \otimes^{p\in\tau(h,b)} \mathsf{f}_{|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}}. \tag{3.11}$$

Proof of Proposition 3.10. Using (3.6) and identifying j_h^b with $|\tau(h,b)|$ we write down

$$\begin{aligned}
&\big[\widehat{g_h^b}\cdot\mathrm{pr}_{j_h^b} : \otimes^{a\in t_h^{-1}b} T^{j^a_{h-1}} sA^a_{h-1} \to T^{j_h^b} sA_h^b\big] \\
&= \big[\otimes^{a\in t_h^{-1}b} T^{|\tau(h-1,a)|} sA^a_{h-1} \xrightarrow{\otimes^{a\in t_h^{-1}b}\Delta_0^{(|\tau(h,b)|)}} \\
&\qquad \bigotimes_{a\in t_h^{-1}b} \bigoplus_{\sum_{p\in\tau(h,b)}|\tau^{-1}_{(h-1,a)}(p)|=|\tau(h-1,a)|} \bigotimes_{p\in\tau(h,b)} T^{|\tau^{-1}_{(h-1,a)}(p)|} sA^a_{h-1} \\
&\qquad \xrightarrow{\sim} \bigoplus_{\sum_{p\in\tau(h,b)}|\tau^{-1}_{(h-1,a)}(p)|=|\tau(h-1,a)|} \bigotimes_{a\in t_h^{-1}b} \bigotimes_{p\in\tau(h,b)} T^{|\tau^{-1}_{(h-1,a)}(p)|} sA^a_{h-1} \\
&\xrightarrow{\oplus\overline{\varkappa}_{t_h^{-1}b,\tau(h,b)}} \bigoplus_{\sum_{p\in\tau(h,b)}|\tau^{-1}_{(h-1,a)}(p)|=|\tau(h-1,a)|} \bigotimes_{p\in\tau(h,b)} \bigotimes_{a\in t_h^{-1}b} T^{|\tau^{-1}_{(h-1,a)}(p)|} sA^a_{h-1} \\
&\qquad\qquad \xrightarrow{\sum\otimes^{p\in\tau(h,b)} g^b_{h,|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}}} \otimes^{p\in\tau(h,b)} sA_h^b = T^{|\tau(h,b)|} sA_h^b\big].
\end{aligned}$$

In the particular case of $j_h^b = |\tau(h,b)| = 1$ we get g_h^b. This is realized for the root $(h,b) = (\max[I],1) = \mathrm{root}_t$ of t, since $\tau(\mathrm{root}_t) = \mathbf{1}$.

Tensor product of these expressions gives a factor of (3.7) for each $h \in I$:

$$\begin{aligned}
&\otimes^{b\in t(h)} \widehat{g_h^b} = \big[\otimes^{a\in t(h-1)} T^{|\tau(h-1,a)|} sA^a_{h-1} \xrightarrow{\lambda^{t_h}} \otimes^{b\in t(h)} \otimes^{a\in t_h^{-1}b} T^{|\tau(h-1,a)|} sA^a_{h-1} \\
&\xrightarrow{\otimes^{b\in t(h)}\otimes^{a\in t_h^{-1}b}\Delta_0^{(|\tau(h,b)|)}} \bigotimes_{b\in t(h)} \bigotimes_{a\in t_h^{-1}b} \bigoplus_{\sum_{p\in\tau(h,b)}|\tau^{-1}_{(h-1,a)}(p)|=|\tau(h-1,a)|} \bigotimes_{p\in\tau(h,b)} T^{|\tau^{-1}_{(h-1,a)}(p)|} sA^a_{h-1} \\
&\quad \xrightarrow{\sim} \bigotimes_{b\in t(h)} \bigoplus_{\sum_{p\in\tau(h,b)}|\tau^{-1}_{(h-1,a)}(p)|=|\tau(h-1,a)|} \bigotimes_{a\in t_h^{-1}b} \bigotimes_{p\in\tau(h,b)} T^{|\tau^{-1}_{(h-1,a)}(p)|} sA^a_{h-1} \\
&\quad \xrightarrow{\otimes^{b\in t(h)}\oplus\overline{\varkappa}_{t_h^{-1}b,\tau(h,b)}} \bigotimes_{b\in t(h)} \bigoplus_{\sum_{p\in\tau(h,b)}|\tau^{-1}_{(h-1,a)}(p)|=|\tau(h-1,a)|} \bigotimes_{p\in\tau(h,b)} \bigotimes_{a\in t_h^{-1}b} T^{|\tau^{-1}_{(h-1,a)}(p)|} sA^a_{h-1} \\
&\qquad \xrightarrow{\otimes^{b\in t(h)}\sum\otimes^{p\in\tau(h,b)} g^b_{h,|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}}} \otimes^{b\in t(h)} \otimes^{p\in\tau(h,b)} sA_h^b = \otimes^{b\in t(h)} T^{|\tau(h,b)|} sA_h^b\big].
\end{aligned}$$

Plugging these expressions into (3.7) we obtain an explicit form of the latter. On the other hand, we compute the image of f_j under the bottom row of (3.8) via the top exterior

path of this diagram

$$
\begin{aligned}
\mathsf{f}_j &= (\otimes^{a\in t(0)}\sigma^{\otimes j^a})\cdot f_j\cdot\sigma^{-1}\\
&\mapsto (\otimes^{a\in t(0)}\sigma^{\otimes j^a})\cdot \sum_{\forall a\in t(0)\ |\tau(0,a)|=j^a}^{t-\text{tree }\tau} \otimes^{h\in I}\otimes^{b\in t(h)}\otimes^{p\in\tau(h,b)} f_{|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}}\cdot\sigma^{-1}\\
&\mapsto (\otimes^{a\in t(0)}\sigma^{\otimes j^a})\cdot \sum_{\forall a\in t(0)\ |\tau(0,a)|=j^a}^{t-\text{tree }\tau} \otimes^{h\in I}\otimes^{b\in t(h)}\otimes^{p\in\tau(h,b)} g^b_{h,|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}}\cdot\sigma^{-1}\\
&\mapsto (\otimes^{a\in t(0)}\sigma^{\otimes j^a})\cdot \mathrm{comp}_{\mathbf{dg}}(t)\Big[\sum_{\forall a\in t(0)\ |\tau(0,a)|=j^a}^{t-\text{tree }\tau} \otimes^{h\in I}\otimes^{b\in t(h)}\otimes^{p\in\tau(h,b)} g^b_{h,|\tau^{-1}_{(h-1,a)}(p)|_{a\in t_h^{-1}b}}\Big]\cdot\sigma^{-1}.
\end{aligned}
$$

The last expression coincides with $\mathrm{comp}(t)(g^b_h)_j$. In fact, multicategory composition in $\underline{\mathbf{dg}}$ restricted to degree 0 cycles g^b_h coincides with the multicategory composition $\mathrm{comp}_{\mathbf{dg}}(t)$ in $\mathbf{dg}$. The latter in our case is the map

$$
\begin{aligned}
&\otimes^{h\in I}\otimes^{b\in t(h)}\otimes^{p\in\tau(h,b)}\mathbf{dg}(\otimes^{a\in t_h^{-1}b}T^{|\tau^{-1}_{(h-1,a)}(p)|}sA^a_{h-1},sA^b_h)\\
&\xrightarrow{\otimes^{h\in I}\otimes^{b\in t(h)}\otimes^{\tau(h,b)}} \otimes^{h\in I}\otimes^{b\in t(h)}\mathbf{dg}(\otimes^{p\in\tau(h,b)}\otimes^{a\in t_h^{-1}b}T^{|\tau^{-1}_{(h-1,a)}(p)|}sA^a_{h-1},\otimes^{p\in\tau(h,b)}sA^b_h)\\
&\xrightarrow{\otimes^{h\in I}\otimes^{b\in t(h)}\mathbf{dg}((\otimes^{a\in t_h^{-1}b}\lambda^{\tau(h-1,a)})\cdot\varkappa_{t_h^{-1}b,\tau(h,b)},1)} \otimes^{h\in I}\otimes^{b\in t(h)}\mathbf{dg}(\otimes^{a\in t_h^{-1}b}T^{|\tau(h-1,a)|}sA^a_{h-1},T^{|\tau(h,b)|}sA^b_h)\\
&\xrightarrow{\otimes^{h\in I}\otimes^{t(h)}} \otimes^{h\in I}\mathbf{dg}(\otimes^{b\in t(h)}\otimes^{a\in t_h^{-1}b}T^{|\tau(h-1,a)|}sA^a_{h-1},\otimes^{b\in t(h)}T^{|\tau(h,b)|}sA^b_h)\\
&\xrightarrow{\otimes^{h\in I}\mathbf{dg}(\lambda^{t_h},1)} \otimes^{h\in I}\mathbf{dg}(\otimes^{a\in t(h-1)}T^{|\tau(h-1,a)|}sA^a_{h-1},\otimes^{b\in t(h)}T^{|\tau(h,b)|}sA^b_h)\\
&\xrightarrow{\mu^I_{\mathbf{dg}}} \mathbf{dg}(\otimes^{a\in t(0)}T^{|\tau(0,a)|}sA^a_0,sA^1_{\max[I]}).
\end{aligned}
$$

Thus, the considered expression coincides with (3.7).

Recall the bijection $\tau\mapsto({}_\psi\tau,(g,c,q)\mapsto{}_{g,c}^{\ q}\tau)$ from (1.11) with the inverse mapping given by (1.12). It is used in formula (1.10) for λ^f. Summation in (3.10) extends precisely over t-trees τ with surjective $\tilde\tau$. Each summand implements an obvious isomorphism $\Bbbk\to\Bbbk^{\otimes-}$. We have

$$
\widetilde{{}_\psi\tau}=\tilde\tau_\psi:[J]\to\mathcal{O}_{\mathsf{sk}}\quad\text{and}\quad\widetilde{{}_{g,c}^{\ q}\tau}=\tilde\tau^{|(c,g,q)}_{[\psi(g-1),\psi(g)]}:[\psi(g)-\psi(g-1)]\to\mathcal{O}_{\mathsf{sk}}.
$$

Therefore, $\tilde\tau$ is surjective iff ${}_\psi\tau$ and all ${}_{g,c}^{\ q}\tau$ are surjective trees. This implies that images of $f_j\in F_{t(0)}$ under both paths in the following diagram

$$
\begin{array}{ccc}
F_{t(0)} & \xrightarrow{\Delta(t_\psi)} & \circledast_{\mathsf{G}}(t_\psi)(F_{t^{-1}_{\psi,g}c})_{(g,c)\in\mathrm{v}(t_\psi)}\\
{\scriptstyle\Delta(t)}\big\downarrow & = & \big\downarrow{\scriptstyle\circledast_{\mathsf{G}}(t_\psi)(\Delta(t^{|c}_{[\psi(g-1),\psi(g)]}))}\\
\circledast_{\mathsf{G}}(t)(F_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)} & \xrightarrow{\lambda^f} & \circledast_{\mathsf{G}}(t_\psi)\big(\circledast_{\mathsf{G}}(t^{|c}_{[\psi(g-1),\psi(g)]})(F_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t^{|c}_{[\psi(g-1),\psi(g)]})}\big)_{(g,c)\in\mathrm{v}(t_\psi)}
\end{array}
\tag{3.12}
$$

are sums indexed by the same set of indices with equal summands. Thus this diagram is commutative and Δ is coassociative. Notice also that the tensor product Δ^{M} over the unit operad $\Bbbk$ coincides with Δ^{G}. $\square$

3.11 Proposition. *Define comultiplication $\Delta^{\mathsf{M}}(t)$ for the A_∞-polymodule $F_\bullet$ and a tree $t:[p]\to\mathcal{O}_{\mathsf{sk}}$, $p>0$, via composition with π from* (1.24)

$$\Delta^{\mathsf{M}}(t)(j)=\bigl[F_{t(0)}(j)\xrightarrow{\Delta^{\mathsf{G}}(t)}\circledast_{\mathsf{G}}(t)(F_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}(j)\xrightarrow{\pi}\circledast_{\mathsf{M}}(t)(F_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}(j)\bigr]$$

(on generators it is still given by (3.10)*). For the only tree $t:[0]\to\mathcal{O}_{\mathsf{sk}}$ define $\Delta^{\mathsf{M}}(t)(j):F_1(j)\to A_\infty(j)$, $f_j\mapsto\delta_{j1}$, $j\geqslant 1$. Then all $\Delta^{\mathsf{M}}(t)(j)$ are chain maps, thus, $(A_\infty,F_\bullet,\Delta^{\mathsf{M}})$ is a* **dg**-*polymodule cooperad.*

Proof. First of all, comultiplication Δ^{M} is coassociative — it satisfies (3.2), since Δ^{G} satisfies (3.12). For trees $t:[1]\to\mathcal{O}_{\mathsf{sk}}$ the morphism $\Delta^{\mathsf{M}}(t)=\Delta^{\mathsf{G}}(t)$ satisfies (3.1).

It suffices to prove the commutation of $\Delta^{\mathsf{M}}(t)$ and ∂ on generators:

$$f_j.\Delta^{\mathsf{M}}(t)\partial=f_j.\partial\Delta^{\mathsf{M}}(t)\text{ for all trees }t:[p]\to\mathcal{O}_{\mathsf{sk}}\text{ and all indices }j\in\mathbb{N}^{t(0)}-0. \tag{3.13}$$

In fact, any element of F_n, $n=|t(0)|$, is a sum of elements of the form $\alpha\bigl((\otimes_{i=1}^n\otimes_{p=1}^{k_1^i+\dots+k_m^i}\omega_p^i)\otimes(\otimes_{r=1}^m f_{k_r})\otimes\omega\bigr)$, where $\omega_p^i\in A_\infty(j_p^i)$, $\omega\in A_\infty(m)$, see the first row of the following diagram

$$\begin{tikzcd}
\Bigl(\bigotimes_{i=1}^{n}\bigotimes_{p=1}^{k_1^i+\dots+k_m^i}A_\infty(j_p^i)\Bigr)\otimes\Bigl(\bigotimes_{r=1}^{m}F_{t(0)}(k_r)\Bigr)\otimes A_\infty(m) \arrow[r, "\alpha"] \arrow[d, "1\otimes(\otimes_{r=1}^m\Delta^{\mathsf{M}}(t))\otimes1"'] & F_{t(0)}\Bigl(\Bigl(\sum_{p=1}^{k_1^i+\dots+k_m^i}j_p^i\Bigr)_{i=1}^n\Bigr) \arrow[d, "\Delta^{\mathsf{M}}(t)"] \\
\Bigl(\bigotimes_{i=1}^{n}\bigotimes_{p=1}^{k_1^i+\dots+k_m^i}A_\infty(j_p^i)\Bigr)\otimes\Bigl(\bigotimes_{r=1}^{m}\circledast_{\mathsf{M}}(t)(F_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}(k_r)\Bigr)\otimes A_\infty(m) \arrow[r, "\alpha"'] & \circledast_{\mathsf{M}}(t)(F_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}\Bigl(\Bigl(\sum_{p=1}^{k_1^i+\dots+k_m^i}j_p^i\Bigr)_{i=1}^n\Bigr)
\end{tikzcd}$$

This square commutes since $\Delta^{\mathsf{M}}(t)$ is a $t(0)\wedge1$-A_∞-module homomorphism of degree 0. Use this square as the top and the bottom faces of a cubical diagram whose vertical edges are given by the differential ∂. We know that α is a chain map. Apply all 3-arrow-paths in this cube to the element $x=(\otimes_{i=1}^n\otimes_{p=1}^{k_1^i+\dots+k_m^i}\omega_p^i)\otimes(\otimes_{r=1}^m f_{k_r})\otimes\omega$ from the top vertex. The equation $x.\bigl(1\otimes(\otimes_{r=1}^m\Delta^{\mathsf{M}}(t))\otimes1\bigr)\partial=x.\partial\bigl(1\otimes(\otimes_{r=1}^m\Delta^{\mathsf{M}}(t))\otimes1\bigr)$ holds by assumption, hence, the equation $x.\alpha\Delta^{\mathsf{M}}(t)\partial=x.\alpha\partial\Delta^{\mathsf{M}}(t)$ holds as well.

For the only tree of height 0, $t : [0] \to \mathcal{O}_{\mathsf{sk}}$, $0 \mapsto t(0) = \mathbf{1}$, and and a positive integer j we have $f_j.\varDelta^{\mathsf{M}}(t)\partial = \delta_{j1}.\partial = 0$. On the other hand,

$$f_j.\partial\varDelta^{\mathsf{M}}(t) = \sum_{r+n+t=j}^{n>1} (1^{\otimes r} \otimes b_n \otimes 1^{\otimes t}) f_{r+1+t}.\varDelta^{\mathsf{M}}(t) - \sum_{i_1+\dots+i_l=j}^{l>1} (f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_l}) b_l.\varDelta^{\mathsf{M}}(t)$$
$$= b_j - b_j = 0$$

by (0.9). For $p = 1$ the map $\varDelta^{\mathsf{M}}(t)$ is the identity morphism. The case of $p > 2$ can be reduced to trees of height smaller than p. In fact, for $\psi : [2] \to [p]$, $0 \mapsto 0$, $1 \mapsto 2$, $2 \mapsto p$, coassociativity equation (3.2) represents $\varDelta^{\mathsf{M}}(t)$ as composition of $\varDelta^{\mathsf{M}}(t_\psi)$ and tensor product of $\varDelta^{\mathsf{M}}(t')$ for trees t' of height 2 or $p - 2$. Therefore, it suffices to prove that $\varDelta^{\mathsf{M}}(t)$ is a chain map for height $p = 2$ of t.

Let us prove equation (3.13) for $t = \{\sqcup_{c=1}^n \mathbf{l}^c \to \mathbf{n} \to \mathbf{1}\}$, $|t(0)| = l^1 + \dots + l^n$. Elements of $\mathbb{N}^{t(0)}$ are written as $j = (j^{c,g} \mid c \in \mathbf{n},\ g \in \mathbf{l}^c)$. Summands of (3.10) are indexed by t-trees $\tau = \tau^1_\lambda$ from (1.29). The tree τ occurs in expansion of $\varDelta^{\mathsf{M}}(t)(f_j)$ if $\sum_{p=1}^{u^c} r_p^{c,g} = j^{c,g}$. Denote $r_p^c = (r_p^{c,g})_{g\in\mathbf{l}^c}$. Thus,

$$f_j.\varDelta^{\mathsf{M}}(t) = \sum_{u\in\mathbb{N}^n} \sum_{\forall c\in\mathbf{n}\ \forall g\in\mathbf{l}^c\ \sum_{p=1}^{u^c} r_p^{c,g}=j^{c,g}} \Big(\bigotimes_{c\in\mathbf{n}} \bigotimes_{p\in\mathbf{u}^c} f_{r_p^c} \Big) \otimes f_u.$$

Equally well we could use $\tau = \tau^1_\rho$ from (1.29), which results in replacing u with k, p with q, and r with s in the above formula. Below we use both presentations. Using the lexicographic order on the set $\sqcup_{c=1}^n \mathbf{u}^c$ ($c < h$ implies that $(c,p) < (h,y)$), we find

$$\begin{aligned} f_j.\varDelta^{\mathsf{M}}(t)\partial = &\sum_{k\in\mathbb{N}^n} \sum_{h\in\mathbf{n}} \sum_{y\in\mathbf{k}^h} \sum_{\forall c\in\mathbf{n}\ \forall g\in\mathbf{l}^c\ \sum_{q=1}^{k^c} s_q^{c,g}=j^{c,g}} \sum_{z\in\mathbf{l}^h} \sum_{a+x+m=s_y^{h,z}} \\ &\Big(\bigotimes^{(c,q)<(h,y)\in\sqcup_b\mathbf{k}^b} f_{s_q^c} \Big) \otimes \lambda^z({}^a1, b_x, {}^m1; f_{s_y^h-(x-1)e_z}) \otimes \Big(\bigotimes^{(c,q)>(h,y)\in\sqcup_b\mathbf{k}^b} f_{s_q^c} \Big) \otimes f_k \end{aligned} \tag{3.14}$$

$$\begin{aligned} - &\sum_{k\in\mathbb{N}^n} \sum_{h\in\mathbf{n}} \sum_{y\in\mathbf{k}^h} \sum_{\forall c\in\mathbf{n}\ \forall g\in\mathbf{l}^c\ \sum_{q=1}^{k^c} s_q^{c,g}=j^{c,g}} \sum_{i_1+\dots+i_w=s_y^h}^{i_1,\dots,i_w\in\mathbb{N}^{l^h}-0} \\ &\Big(\bigotimes^{(c,q)<(h,y)\in\sqcup_b\mathbf{k}^b} f_{s_q^c} \Big) \otimes \rho((f_{i_s})_{s=1}^w; b_w) \otimes \Big(\bigotimes^{(c,q)>(h,y)\in\sqcup_b\mathbf{k}^b} f_{s_q^c} \Big) \otimes f_k \end{aligned} \tag{3.15}$$

$$+ \sum_{u\in\mathbb{N}^n} \sum_{\forall c\in\mathbf{n}\ \forall g\in\mathbf{l}^c\ \sum_{p=1}^{u^c} r_p^{c,g}=j^{c,g}} \sum_{h\in\mathbf{n}} \sum_{a+w+m=u^h} \Big(\bigotimes_{c\in\mathbf{n}} \bigotimes_{p\in\mathbf{u}^c} f_{r_p^c} \Big) \otimes \lambda^h({}^a1, b_w, {}^m1; f_{u-(w-1)e_h}) \tag{3.16}$$

$$- \sum_{u\in\mathbb{N}^n} \sum_{\forall c\in\mathbf{n}\ \forall g\in\mathbf{l}^c\ \sum_{p=1}^{u^c} r_p^{c,g}=j^{c,g}} \sum_{u_1+\dots+u_w=u}^{u_1,\dots,u_w\in\mathbb{N}^n-0} \Big(\bigotimes_{c\in\mathbf{n}} \bigotimes_{p\in\mathbf{u}^c} f_{r_p^c} \Big) \otimes \rho((f_{u_v})_{v=1}^w; b_w). \tag{3.17}$$

Sums (3.15) and (3.16) cancel each other. In fact, consider the tree $\tau^1_+ : t^1_+ \to \mathcal{O}_{\mathsf{sk}}$ from (1.28) with $j^c_q = 1$ for $c \neq h$, $j^h_q = 1$ for $q \neq y$, and $j^h_y = w$. This implies relations $u^c = k^c$ for $c \neq h$, $u^h = k^h + w - 1$, $s^{c,g}_q = r^{c,g}_q$ for $c \neq h$, $s^{h,g}_q = r^{h,g}_q$ if $q < y$, $s^{h,g}_y = \sum_{p=y}^{y+w-1} r^{h,g}_p$, and $s^{h,g}_q = r^{h,g}_{q+w-1}$ if $q > y$. The images of the element

$$\Big(\bigotimes^{(c,q)<(h,y)} f_{s^c_q}\Big) \otimes \Big(\bigotimes^{v\in\mathbf{w}} f_{i_v}\Big) \otimes \Big(\bigotimes^{(c,q)>(h,y)} f_{s^c_q}\Big) \otimes \Big(\bigotimes^{(c,q)<(h,y)} 1\Big) \otimes b_w \otimes \Big(\bigotimes^{(c,q)>(h,y)} 1\Big) \otimes f_k$$
$$= \Big(\bigotimes^{c\in\mathbf{n}} \bigotimes^{p\in\mathbf{u}^c} f_{r^c_p}\Big) \otimes \Big(\bigotimes^{(c,q)<(h,y)\in\sqcup_b \mathbf{k}^b} 1\Big) \otimes b_w \otimes \Big(\bigotimes^{(c,q)>(h,y)\in\sqcup_b \mathbf{k}^b} 1\Big) \otimes f_k,$$

where $i_v = r^{h,g}_{y,v} = r^{h,g}_p$ for $p = y - 1 + v$, $v \in \mathbf{w}$, under maps (1.31) and (1.30) are, respectively,

$$\Big(\bigotimes^{(c,q)<(h,y)\in\sqcup_b \mathbf{k}^b} f_{s^c_q}\Big) \otimes \rho((f_{i_v})^w_{v=1}; b_w) \otimes \Big(\bigotimes^{(c,q)>(h,y)\in\sqcup_b \mathbf{k}^b} f_{s^c_q}\Big) \otimes f_k,$$
$$\Big(\bigotimes^{c\in\mathbf{n}} \bigotimes^{p\in\mathbf{u}^c} f_{r^c_p}\Big) \otimes \lambda^h({}^{y-1}1, b_w, {}^{k^h-y}1; f_k).$$

These are identified with summands of sums (3.15) and (3.16). Namely, we identify a with $y-1$, m with $k^h - y$ and notice that $u - (w-1)e_h = k$. Therefore, summands of these sums pairwise cancel each other.

We claim that the difference of sums (3.14) and (3.17) equals to $f_j.\partial\Delta^{\mathsf{M}}(t)$. In fact, by (2.6)

$$f_j.\partial\Delta^{\mathsf{M}}(t) = \sum_{h=1}^{n}\sum_{z=1}^{l^h}\sum_{\gamma+x+m=j^{h,z}}^{x>1} \lambda^{h,z}({}^{\gamma}1, b_x, {}^{m}1; f_{j-(x-1)e_{h,z}}.\Delta^{\mathsf{M}}(t))$$
$$- \sum_{j_1+\dots+j_w=j}^{j_1,\dots,j_w\in\mathbb{N}^{t(0)}-0} \rho((f_{j_v}.\Delta^{\mathsf{M}}(t))^w_{v=1}; b_w)$$
$$= \sum_{h=1}^{n}\sum_{z=1}^{l^h}\sum_{\gamma+x+m=j^{h,z}}^{x>1}\sum_{k\in\mathbb{N}^n}\sum_{\forall c\in\mathbf{n}\ \forall g\in\mathbf{l}^c\ \sum_{q=1}^{k^c} s^{c,g}_q = j^{c,g}-(x-1)\delta^c_h\delta^g_z} \lambda^{h,z}\Big({}^{\gamma}1, b_x, {}^{m}1; \Big(\bigotimes_{c\in\mathbf{n}}\bigotimes_{q\in\mathbf{k}^c} f_{s^c_q}\Big)\otimes f_k\Big) \tag{3.18}$$
$$- \sum_{j_1+\dots+j_w=j}^{j_1,\dots,j_w\in\mathbb{N}^{t(0)}-0} \rho\Bigg(\Bigg(\sum_{u_v\in\mathbb{N}^n}\sum_{\forall c\in\mathbf{n}\ \forall g\in\mathbf{l}^c\ \sum_{p=u^c_1+\dots+u^c_{v-1}+1}^{u^c_1+\dots+u^c_{v-1}+u^c_v} r^{c,g}_p = j^{c,g}_v} \Big(\bigotimes^{c\in\mathbf{n}} \bigotimes_{p=1+\sum_{\alpha=1}^{v-1}u^c_\alpha}^{\sum_{\alpha=1}^{v}u^c_\alpha} f_{r^c_p}\Big)\otimes f_{u_v}\Bigg)^w_{v=1}; b_w\Bigg). \tag{3.19}$$

Sum (3.19) coincides with (3.17). Sums (3.14) and (3.18) are equal after the following identification. The sum in (3.18) over $\gamma \in \mathbb{N}$ such that $1 \leqslant \gamma + 1 \leqslant j^{h,z} - x + 1 = \sum_{q=1}^{k^h} s^{h,z}_q$

is equivalent to summing over pairs $(y, \ddot{\imath})$, $y \in \mathbf{k}^h$, $\ddot{\imath} \in \mathbf{s}_y^{h,z}$, namely, $\gamma + 1 = \ddot{\imath} + \sum_{q=1}^{y-1} s_q^{h,z}$. The expression $\ddot{\imath} - 1$ is denoted a in (3.14). Variables $s_q^{c,g}$ coincide in both expressions, except $s_y^{h,z}$ whose values differ by $x - 1$. This finishes the proof. $\square$

3.12 Definition. A polymodule cooperad morphism $(h, p) : (\mathcal{A}, F) \to (\mathcal{B}, G)$ of degree 1 is a family of $n \wedge 1$-operad module homomorphisms of degree 1

$$({}^n h; p_n; h) : ({}^n\mathcal{A}; F_n; \mathcal{A}) \to ({}^n\mathcal{B}; G_n; \mathcal{B})$$

such that for all non-decreasing maps $\phi : I \to J$, the induced $\psi = [\phi] : [J] \to [I]$ as in (0.12), and for all trees $t : [I] \to \mathcal{O}_{\mathsf{sk}}$ each wall (vertical face) of cube at Fig. 1 commutes up to prescribed sign, varying with the summand of $\circledast_{\mathsf{M}}$. Here the sign $(-1)^{s(f)}$ comes from the permutation of factors $p(k)$ of degree $1 - |k|$ according to rule (1.10). The floor and the ceiling of cube at Fig. 1 commute.

3.13 Remark. For each summand of the target of cube at Fig. 1 (the right lower vertex) the product of all signs on walls of this diagram is $+1$. Indeed, the same signs occur in diagram

$$\begin{tikzcd}[column sep=small]
H'({}^{<}t) \arrow[rr, "\mathrm{comp}(t_\psi)"] \arrow[dr, "\mathrm{comp}(t)"] \arrow[dd, "h({}^{<}t)"'] & \arrow[d, phantom, "(-1)^{c(\widetilde{\psi\tau})}"] & \circledast_{\mathsf{G}}(t_\psi)H'(t_\psi) \arrow[dr, "\circledast_{\mathsf{G}}(t_\psi)(\mathrm{comp}(t^{|c}_{[\psi(g-1),\psi(g)]}))"] \arrow[dd, "\circledast_{\mathsf{G}}(t_\psi)h(t_\psi)" near end] \arrow[d, phantom, "(-1)^{\sum_{g,c,q} c(\widetilde{{}_{g,c}^{\;q}\tau})}"'] & \\
& \circledast_{\mathsf{G}}(t)H'(t) \arrow[rr, "\lambda^f", "\sim"' near start] \arrow[dd, "\circledast_{\mathsf{G}}(t_\psi)h(t)"' near start] & \, & \circledast_{\mathsf{G}}(t_\psi)\big(\circledast_{\mathsf{G}}(t^{|c}_{[\psi(g-1),\psi(g)]})H'(t^{|c}_{[\psi(g-1),\psi(g)]})\big)_{(g,c)\in\mathrm{v}(t_\psi)} \arrow[dd, "\circledast_{\mathsf{G}}(t_\psi)(\circledast_{\mathsf{G}}(t^{|c}_{[\psi(g-1),\psi(g)]})h(t^{|c}_{[\psi(g-1),\psi(g)]}))"] \\
H({}^{<}t) \arrow[rr, "\mathrm{comp}(t_\psi)" near end] \arrow[dr, "\mathrm{comp}(t)"'] & \arrow[d, phantom, "(-1)^{c(\tilde{\tau})}"] & \circledast_{\mathsf{G}}(t_\psi)H(t_\psi) \arrow[dr, "\circledast_{\mathsf{G}}(t_\psi)(\mathrm{comp}(t^{|c}_{[\psi(g-1),\psi(g)]}))"'] \arrow[d, phantom, "(-1)^{s(f)}"'] & \\
& \circledast_{\mathsf{G}}(t)H(t) \arrow[rr, "\lambda^f", "\sim"'] & \, & \circledast_{\mathsf{G}}(t_\psi)\big(\circledast_{\mathsf{G}}(t^{|c}_{[\psi(g-1),\psi(g)]})H(t^{|c}_{[\psi(g-1),\psi(g)]})\big)_{(g,c)\in\mathrm{v}(t_\psi)}
\end{tikzcd} \tag{3.20}$$

relating composition comp for functors *hom* and the shifts σ. The following abbreviations for families of operad polymodules and their degree 1 homomorphisms are used in diagram (3.20):

$$\begin{aligned}
H'(t) &= \big(hom((sA^a_{h-1})_{a\in t_h^{-1}b}; sA^b_h)\big)_{(h,b)\in\mathrm{v}(t)}, \\
h(t) &= \big(hom((\sigma)_{a\in t_h^{-1}b}; \sigma^{-1})\big)_{(h,b)\in\mathrm{v}(t)}, \\
H(t) &= \big(hom((A^a_{h-1})_{a\in t_h^{-1}b}; A^b_h)\big)_{(h,b)\in\mathrm{v}(t)}.
\end{aligned}$$

$$
\begin{tikzcd}
F_{t(0)} \arrow[rr, "\Delta(t_\psi)"] \arrow[dr, "\Delta(t)"] \arrow[dd, "p"'] \arrow[r, phantom, "(-1)^{c(\widetilde{\psi\tau})}"] & & \circledast_{\mathsf{M}}(t_\psi)\big(F_{t^{-1}_{\psi,g}c}\big)_{(g,c)\in\mathrm{v}(t_\psi)} \arrow[dr, "\circledast_{\mathsf{M}}(t_\psi)(\Delta(t^{|c}_{[\psi(g-1),\psi(g)]}))"] \arrow[dd, "\circledast_{\mathsf{M}}(t_\psi)(p)" near end] \arrow[d, phantom, "(-1)^{\sum_{g,c,q} c(\overset{q}{\widetilde{g,c}}\,\tau)}"] & \\
& \circledast_{\mathsf{M}}(t)\big(F_{t_h^{-1}b}\big)_{(h,b)\in\mathrm{v}(t)} \arrow[rr, "\lambda^f", "\sim"'] \arrow[dd, "\circledast_{\mathsf{M}}(t_\psi)(p)"' near start] \arrow[dr, phantom, "(-1)^{s(f)}"] & & \circledast_{\mathsf{M}}(t_\psi)\Big(\circledast_{\mathsf{M}}(t^{|c}_{[\psi(g-1),\psi(g)]})\big(F_{t_h^{-1}b}\big)_{(h,b)\in\mathrm{v}(t^{|c}_{[\psi(g-1),\psi(g)]})}\Big)_{(g,c)\in\mathrm{v}(t_\psi)} \arrow[dd, "\circledast_{\mathsf{M}}(t_\psi)(\circledast_{\mathsf{M}}(t^{|c}_{[\psi(g-1),\psi(g)]})(p))"] \\
G_{t(0)} \arrow[rr, "\Delta(t_\psi)" near end] \arrow[dr, "\Delta(t)"'] \arrow[r, phantom, "(-1)^{c(\widetilde{\tau})}"] & & \circledast_{\mathsf{M}}(t_\psi)\big(G_{t^{-1}_{\psi,g}c}\big)_{(g,c)\in\mathrm{v}(t_\psi)} \arrow[dr, "\circledast_{\mathsf{M}}(t_\psi)(\Delta(t^{|c}_{[\psi(g-1),\psi(g)]}))"'] & \\
& \circledast_{\mathsf{M}}(t)\big(G_{t_h^{-1}b}\big)_{(h,b)\in\mathrm{v}(t)} \arrow[rr, "\lambda^f", "\sim"'] & & \circledast_{\mathsf{M}}(t_\psi)\Big(\circledast_{\mathsf{M}}(t^{|c}_{[\psi(g-1),\psi(g)]})\big(G_{t_h^{-1}b}\big)_{(h,b)\in\mathrm{v}(t^{|c}_{[\psi(g-1),\psi(g)]})}\Big)_{(g,c)\in\mathrm{v}(t_\psi)}
\end{tikzcd}
$$

Figure 1: Cube equations for a polymodule cooperad morphism of degree 1

For $t : [I] \to \mathcal{O}_{\mathsf{sk}}$ the tree ${}^{<}t$ means the corolla $t(0) \to \mathbf{1} = t_{\max[I]}$ labelled with $(A_0^a)_{a\in t(0)}$ and $A^1_{\max[I]}$. Thus $H({}^{<}t) = \mathit{hom}((A_0^a)_{a\in t(0)}; A^1_{\max[I]})$ etc. The commuting floor of diagram (3.20) is precisely equation (1.20). The commuting ceiling is similar, with sA_h^b instead of A_h^b. Commutativity of the floor and the ceiling implies the claim on signs.

3.14 Lemma. *Let $({}^n h; p_n; h) : ({}^n\mathcal{A}; F_n; \mathcal{A}) \to ({}^n\mathcal{B}; G_n; \mathcal{B})$, $n \geqslant 0$, be a family of invertible $n \wedge 1$-operad module homomorphisms of degree 1. If the family $(\mathcal{A}, F_\bullet)$ or $(\mathcal{B}, G_\bullet)$ has a structure of a polymodule cooperad, then the other family has a unique cooperad structure such that $(h, p_\bullet)$ is a degree 1 isomorphism of cooperads.*

Proof. Follows from diagram at Fig. 1 and the equation between signs on walls proven in Remark 3.13. $\square$

3.15 Corollary. *The family of $n \wedge 1$-operad modules $(\mathrm{A}_\infty, \mathrm{F}_n)$, $n \geqslant 0$, equipped with comultiplication* (3.11) *is a polymodule cooperad.*

3.16. Comultiplication for homotopy unital case. Let multiquiver $\mathsf{a}_\infty^{hu} = \mathsf{H}$ be convolution of $\mathrm{F}^{hu} : \mathsf{F} \to \mathsf{M}$ and $\mathcal{H}om : \mathsf{B} \to \mathsf{M}$ coming from $\underline{\mathsf{C}}_{\Bbbk}$. Objects of a_∞^{hu} are homotopy unital A_∞-algebras and morphisms are homotopy unital A_∞-morphisms.

There is a multiquiver map $\text{-}^+ : \mathsf{a}_\infty^{hu} \to \mathsf{a}_\infty$, $(A, \mathsf{i}, m_1, m_{n_1;n_2;\dots;n_k} \mid k + \sum_{q=1}^k n_q \geqslant 3) \to (A^+, m_n^+ \mid n \geqslant 1)$, where $A^+ = A \oplus \Bbbk \mathsf{1}^{\mathsf{su}} \oplus \Bbbk \mathsf{j}$ is strictly unital with the strict unit $\mathsf{1}^{\mathsf{su}}$, $m_n^+\big|_{A^{\otimes n}} = m_n$, $\mathsf{j}m_1^+ = \mathsf{1}^{\mathsf{su}} - \mathsf{i}$ and

$$(1^{\otimes n_1} \otimes \mathsf{j} \otimes 1^{\otimes n_2} \otimes \mathsf{j} \otimes \dots \otimes 1^{\otimes n_{k-1}} \otimes \mathsf{j} \otimes 1^{\otimes n_k}) m^+_{n+k-1} = m_{n_1;n_2;\dots;n_k} : A^{\otimes n+k-1} \to A$$

for $k \geqslant 1$, $n_q \geqslant 0$, $n = \sum_{q=1}^k n_q$, $n + k \geqslant 3$. On morphisms with n arguments we have

$$\mathsf{f} = (\mathsf{v}_k, \mathsf{f}_{(\ell_1^k;\ell_2^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}}) \mapsto \mathsf{f}^+ = (\mathsf{f}_j^+ \mid j \in \mathbb{N}^n - 0),$$

where $\mathsf{j}\mathsf{f}^+_{e_k} = \mathsf{v}_k + \mathsf{j}\rho_\varnothing$ and for $|\hat\ell| \geqslant 2$

$$\bigl[\otimes^{k\in\mathbf{n}} T^{\ell^k} A_k \xrightarrow{\otimes^{k\in\mathbf{n}}(1^{\otimes\ell_1^k}\otimes\mathsf{j}\otimes1^{\otimes\ell_2^k}\otimes\mathsf{j}\otimes\dots\otimes1^{\otimes\ell^k_{t^k-1}}\otimes\mathsf{j}\otimes1^{\otimes\ell^k_{t^k}})} \otimes^{k\in\mathbf{n}} T^{\hat\ell^k} A_k^+ \xrightarrow{\mathsf{f}^+_{\hat\ell}} B\bigr]$$
$$= \lambda_{\hat\ell}\bigl(({}^{\ell_1^k}1, \mathsf{j}, {}^{\ell_2^k}1, \mathsf{j}, \dots, {}^{\ell^k_{t^k-1}}1, \mathsf{j}, {}^{\ell^k_{t^k}}1)_{k\in\mathbf{n}}; \mathsf{f}^+_{\hat\ell}\bigr) = \mathsf{f}_{(\ell_1^k;\ell_2^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}}.$$

This multiquiver map is injective on morphisms and the conditions of Definition 2.20 describe its image. The image is closed under composition in a_∞, hence, it is a sub-multicategory. In this way a_∞^{hu} becomes a multicategory and $\text{-}^+ : \mathsf{a}_\infty^{hu} \to \mathsf{a}_\infty$ becomes a multifunctor. Composing it with the multifunctor $Ts : \mathsf{a}_\infty \to \mathbf{dgac}$ we get again a full and faithful embedding $Ts(\text{-})^+ : \mathsf{a}_\infty^{hu} \to \mathbf{dgac}$. Its image is described by conditions parallel to that of Definition 2.20:

(1) f^+ is a strictly unital;

(2) $\widehat{\mathsf{f}^+}(1\otimes\cdots\otimes 1\otimes(A_k+\mathsf{j}^{A_k})\otimes 1\otimes\cdots\otimes 1)\subset B+\mathsf{j}^B$;

(3) $\widehat{\mathsf{f}^+}(\otimes^{k\in\mathbf{n}}TA_k)\subset TB$;

(4) $\widehat{\mathsf{f}^+}(\otimes^{k\in\mathbf{n}}T^{\ell^k}(A_k\oplus\Bbbk\mathsf{j}^{A_k}))\subset B\oplus T^{>1}(B\oplus\Bbbk\mathsf{j}^B)$ for each $\ell\in\mathbb{N}^n$, $|\ell|>1$.

One checks directly that the set of such coalgebra morphisms is closed under composition.

3.16.1. Actions of operads in the tensor product of operad modules. Assume given a tree $t:[l]\to\mathcal{O}_{\mathsf{sk}}$ and a family $(\mathcal{P}^b_h)_{(h,b)\in\mathrm{v}(t)}$ of $t_h^{-1}b\wedge 1$-operad modules. Their tensor product $\mathcal{P}=\circledast(t)(\mathcal{P}^b_h)_{(h,b)\in\mathrm{v}(t)}$ is a $t(0)\wedge 1$-operad module. Operads acting on the left are $(\mathcal{A}_a)_{a\in t(0)}$, where $(\mathcal{A}_a)_{a\in t_1^{-1}b}$ act on $\mathcal{P}^b_1$ for $b\in t(1)$. The summand indexed by t-tree τ, $\tau(0,a)=k^a$ is mapped by λ from (1.18) to the t-tree $\bar\tau$ obtained as follows. There is a tree $t':[-1,l]\cap\mathbb{Z}\to\mathcal{O}_{\mathsf{sk}}$, $t'\big|_{[l]}=t$, $t'_0=\mathrm{id}_{t(0)}$, and a t'-tree τ' such that $\tau'\big|_{[l]}=\tau$, and the maps $\tau'_{(-1,a)}$ correspond to partition $\sum_{p=1}^{k^a}j^a_p$ into k^a summands for $a\in t(0)$. The t-tree $\bar\tau$ is obtained from τ' by dropping the intermediate level 0, so that level -1 becomes level 0. The mapping in question is the tensor product over $b\in t(1)$ of actions λ of $(\mathcal{A}_a)_{a\in t_1^{-1}b}$ on $\mathcal{P}^b_1$.

The operad acting on the right of $\mathcal{P}$ is the operad $\mathcal{B}$ acting on the right of $\mathcal{P}^1_l$. The action map ρ given by (1.18) sends the summand $\big(\otimes_{r=1}^m\mathcal{P}(\tau_r)\big)\otimes\mathcal{B}(m)\to\mathcal{P}(\tau')$, where τ' and τ_r, $1\leqslant r\leqslant m$, are t-trees, $\mathcal{P}=\oplus_{t-\text{tree }\tau}\mathcal{P}(\tau)$, and τ' is constructed as $\tau_1\sqcup\tau_2\sqcup\cdots\sqcup\tau_m$ with identified roots. The action map ρ for $\mathcal{P}$ is the identity map tensored with the action ρ for $\mathcal{P}^1_l$.

In particular, the map $\rho^{\mathcal{P}}_\varnothing:\mathcal{B}(0)\to\mathcal{P}(0)$ sends $\mathcal{B}(0)$ to the summand $\mathcal{P}(\tau_0)\simeq\mathcal{P}^1_l(0)$ via $\rho^{\mathcal{P}^1_l}_\varnothing:\mathcal{B}(0)\to\mathcal{P}^1_l(0)$, where $\tau_0(h,b)=\varnothing$ for $(h,b)\in\mathrm{v}(t)-\{\mathrm{root}\}$, while $\tau_0(\mathrm{root})=\tau_0(l,1)=\mathbf{1}$.

Comultiplication (3.11) extends in a unique way to $(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_n)$, which differs from $(\mathrm{A}_\infty,\mathrm{F}_n)$ by a direct summand $(\Bbbk 1^{\mathsf{su}},\Bbbk 1^{\mathsf{su}}\rho_\varnothing)$, see (2.30). In fact, for a tree $t:[I]\to\mathcal{O}_{\mathsf{sk}}$ the equation

$$\rho_\varnothing=\big[\mathrm{A}^{\mathsf{su}}_\infty(0)\xrightarrow[\sim]{\rho_\varnothing}\mathrm{F}^{\mathsf{su}}_{t(0)}(0)\xrightarrow{\Delta(t)}\circledast_{\mathsf{M}}(t)(\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}(0)\big]\tag{3.21}$$

is one of those saying that $\Delta(t)$ agree with ρ (see (2.2) with $l=0$). So we set $\Delta(t)\big(\rho_\varnothing(1^{\mathsf{su}})\big)=\rho_\varnothing(1^{\mathsf{su}})$. For non-empty I the image of (3.21) is contained in the image of the summand $\mathrm{F}^{\mathsf{su}}_{t(|I|-1)}(0)$ of $\circledast_{\mathsf{G}}(t)(\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}(0)$ indexed by the t-tree τ with $\tau(h,b)=\varnothing$ for all (h,b) such that $0\leqslant h<|I|$, while $\tau(|I|,1)=\mathbf{1}$. For the tree $t:[0]\to\mathcal{O}_{\mathsf{sk}}$ (3.21) is the right action in the regular bimodule $\mathrm{A}^{\mathsf{su}}_\infty$:

$$\mathrm{id}=\rho_\varnothing=\big[\mathrm{A}^{\mathsf{su}}_\infty(0)\xrightarrow[\sim]{\rho_\varnothing}\mathrm{F}^{\mathsf{su}}_1(0)\xrightarrow{\Delta(t)}\mathrm{A}^{\mathsf{su}}_\infty(0)\big].$$

We can be more precise in this case: $\Delta(t)\big(\rho_\varnothing(1^{\mathsf{su}})\big)=\rho_\varnothing(1^{\mathsf{su}})=1^{\mathsf{su}}$.

So extended comultiplication obviously agrees with the left action λ (see (2.4) with $k=0$). It agrees also with the right action ρ, see (2.2) for $l>0$ with $J=\{q\in\mathbf{l}\mid k_q=0\}$. We may take elements $1^{\mathsf{su}}\rho_{\varnothing}$ in each place $\mathcal{P}(0)=\mathrm{F}^{\mathsf{su}}_n(0)$ for $q\in J$. Then $1^{\mathsf{su}}\rho_{\varnothing}$ will appear also in $\mathcal{Q}(0)=\circledast_{\mathsf{M}}(t)(\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}(0)$ for the same q. Using associativity of ρ we can absorb those 1^{su} into an element of $\mathrm{A}^{\mathsf{su}}_\infty$ and get rid of 1^{su}'s completely. The equation is reduced to the case of $(\mathrm{A}_\infty,\mathrm{F}_n)$, which is already verified. Coassociativity of extended comultiplication is obvious.

Let us extend comultiplication further to

$$(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_n)\langle\mathsf{i},\mathsf{j}\rangle\simeq\big(\mathrm{A}^{\mathsf{su}}_\infty\langle\mathsf{i},\mathsf{j}\rangle,\bigcirc_{k=0}^n\mathrm{A}^{\mathsf{su}}_\infty\langle\mathsf{i},\mathsf{j}\rangle\odot^k_{\mathrm{A}^{\mathsf{su}}_\infty}\mathrm{F}^{\mathsf{su}}_n\big)$$

using Proposition 1.36. Let $n=|t(0)|$. The comultiplication is the lower diagonal in

$$\begin{tikzcd}
\circledast_{\mathsf{M}}(t)(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)} \arrow[r, hook] & \circledast_{\mathsf{M}}(t)(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}\langle\mathsf{i},\mathsf{j}\rangle \arrow[d, "\wr"] \\
(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_n) \arrow[u, "\Delta^{\mathsf{M}}(t)"] \arrow[d, hook] & \big(\mathrm{A}^{\mathsf{su}}_\infty\langle\mathsf{i},\mathsf{j}\rangle,\bigcirc_{k=0}^n\mathrm{A}^{\mathsf{su}}_\infty\langle\mathsf{i},\mathsf{j}\rangle\odot^k_{\mathrm{A}^{\mathsf{su}}_\infty}\circledast_{\mathsf{M}}(t)(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)}\big) \arrow[d, hook] \\
(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_n)\langle\mathsf{i},\mathsf{j}\rangle \arrow[ur, "\Delta^{\mathsf{M}}(t)\langle\mathsf{i},\mathsf{j}\rangle"] \arrow[dr, "\Delta^{\mathsf{M}}(t)"'] & \circledast_{\mathsf{M}}(t)(\mathrm{A}^{\mathsf{su}}_\infty\langle\mathsf{i},\mathsf{j}\rangle,\bigcirc_{k\in[t_h^{-1}b]}\mathrm{A}^{\mathsf{su}}_\infty\langle\mathsf{i},\mathsf{j}\rangle\odot^k_{\mathrm{A}^{\mathsf{su}}_\infty}\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b})_{(h,b)\in\mathrm{v}(t)} \arrow[d, "\wr"] \\
 & \circledast_{\mathsf{M}}(t)(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b})\langle\mathsf{i},\mathsf{j}\rangle_{(h,b)\in\mathrm{v}(t)}
\end{tikzcd}$$

Proof of coassociativity is contained in the following diagram. The operad module $(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_n)$ is short-handed to $\mathrm{F}^{\mathsf{su}}_n$. Similarly $\mathrm{F}^{\mathsf{su}}_n\langle\mathsf{i},\mathsf{j}\rangle$ stands for $(\mathrm{A}^{\mathsf{su}}_\infty,\mathrm{F}^{\mathsf{su}}_n)\langle\mathsf{i},\mathsf{j}\rangle$. Being diagram (3.2) the top square commutes. The middle square parallel to the top face is obtained by adding freely operations i and j. Hence it also commutes. The vertical faces

commute as well, therefore, the bottom quadrangle is commutative.

$$
\begin{tikzcd}
& \circledast^{(h,b)}_{\mathsf{M}}(t)\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b} \arrow[rr, "\lambda^f"] \arrow[dd, hook] & & \circledast^{(g,c)}_{\mathsf{M}}(t_\psi)\big(\circledast^{(h,b)}_{\mathsf{M}}(t^{|c}_{[\psi(g-1),\psi(g)]})\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b}\big) \arrow[dd] \\
\mathrm{F}^{\mathsf{su}}_{t(0)} \arrow[ur, "\Delta(t)"] \arrow[rr, "\Delta(t_\psi)"] \arrow[dd, hook] & & \circledast^{(g,c)}_{\mathsf{M}}(t_\psi)\mathrm{F}_{t^{-1}_{\psi,g}c} \arrow[ur, "\circledast^{(g,c)}_{\mathsf{M}}(t_\psi)(\Delta(t^{|c}_{[\psi(g-1),\psi(g)]}))"'] \arrow[dd, hook] & \\
& {[\circledast^{(h,b)}_{\mathsf{M}}(t)\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b}]\langle \mathsf{i},\mathsf{j}\rangle} \arrow[rr, "\lambda^f\langle \mathsf{i},\mathsf{j}\rangle"] \arrow[dddd, hook] & & \big[\circledast^{(g,c)}_{\mathsf{M}}(t_\psi)\big(\circledast^{(h,b)}_{\mathsf{M}}(t^{|c}_{[\psi(g-1),\psi(g)]})\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b}\big)\big]\langle \mathsf{i},\mathsf{j}\rangle \arrow[dd] \\
\mathrm{F}^{\mathsf{su}}_{t(0)}\langle \mathsf{i},\mathsf{j}\rangle \arrow[ur, "\Delta(t)\langle \mathsf{i},\mathsf{j}\rangle"] \arrow[rr, "\Delta(t_\psi)\langle \mathsf{i},\mathsf{j}\rangle"] \arrow[ddrr, "\Delta(t_\psi)"] \arrow[dddr, "\Delta(t)"'] & & {[\circledast^{(g,c)}_{\mathsf{M}}(t_\psi)\mathrm{F}_{t^{-1}_{\psi,g}c}]\langle \mathsf{i},\mathsf{j}\rangle} \arrow[ur, "{[\circledast^{(g,c)}_{\mathsf{M}}(t_\psi)(\Delta(t^{|c}_{[\psi(g-1),\psi(g)]}))]\langle \mathsf{i},\mathsf{j}\rangle}"'] \arrow[dd, hook] & \\
& & & \circledast^{(g,c)}_{\mathsf{M}}(t_\psi)\big[\big(\circledast^{(h,b)}_{\mathsf{M}}(t^{|c}_{[\psi(g-1),\psi(g)]})\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b}\big)\langle \mathsf{i},\mathsf{j}\rangle\big] \arrow[dd, hook] \\
& & \circledast^{(g,c)}_{\mathsf{M}}(t_\psi)[\mathrm{F}_{t^{-1}_{\psi,g}c}\langle \mathsf{i},\mathsf{j}\rangle] \arrow[ur, "{\circledast^{(g,c)}_{\mathsf{M}}(t_\psi)[(\Delta(t^{|c}_{[\psi(g-1),\psi(g)]}))\langle \mathsf{i},\mathsf{j}\rangle]}"'] \arrow[dr, "\circledast^{(g,c)}_{\mathsf{M}}(t_\psi)(\Delta(t^{|c}_{[\psi(g-1),\psi(g)]}))"] & \\
& \circledast^{(h,b)}_{\mathsf{M}}(t)[\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b}\langle \mathsf{i},\mathsf{j}\rangle] \arrow[rr, "\lambda^f"] & & \circledast^{(g,c)}_{\mathsf{M}}(t_\psi)\big(\circledast^{(h,b)}_{\mathsf{M}}(t^{|c}_{[\psi(g-1),\psi(g)]})\big)\big[\mathrm{F}^{\mathsf{su}}_{t_h^{-1}b}\langle \mathsf{i},\mathsf{j}\rangle\big]\big)
\end{tikzcd}
$$

Thus, a polymodule cooperad $(\mathrm{A}^{\mathsf{su}}_\infty, \mathrm{F}^{\mathsf{su}})\langle \mathsf{i},\mathsf{j}\rangle$ is constructed. By Lemma 3.14 there is a polymodule cooperad $(A^{\mathsf{su}}_\infty, F^{\mathsf{su}})\langle \mathbf{i},\mathbf{j}\rangle$ isomorphic to it via a degree 1 isomorphism.

3.17 Proposition. *The collection of operad submodules* $(A^{hu}_\infty, F^{hu}) \subset (A^{\mathsf{su}}_\infty, F^{\mathsf{su}})\langle \mathbf{i},\mathbf{j}\rangle$ *is a subcooperad.*

Proof. Assume that $k \in t(0)$ for a l-tree t. Let us compute $\varDelta^{\mathsf{M}}(t)(f_{e_k})$. Notice that there exists the only t-tree τ such that $\tilde{\tau}$ is surjective and $|\tau(0,a)| = \delta^a_k$ for all $a \in t(0)$. In fact, $\tilde{\tau}(0) = \mathbf{1}$, hence, $\tilde{\tau}(h) = \mathbf{1}$ for all $h \in [I]$. The tree τ is given by the formula

$$\tau(h,b) = \begin{cases} \mathbf{1}, & \text{if } b = t_h \dots t_2 t_1(k), \\ \varnothing, & \text{otherwise.} \end{cases}$$

Denoting $e^S_p \in \mathbb{N}^S$ a basis vector for $p \in S$ we find

$$\varDelta^{\mathsf{M}}(t)(f_{e^{t(0)}_k}) = f_{e^{t_1}_k t_1^{-1} t_1 k} \otimes f_{e^{t_2}_{t_1 k} t_2^{-1} t_2 t_1 k} \otimes f_{e^{t_3}_{t_2 t_1 k} t_3^{-1} t_3 t_2 t_1 k} \otimes \cdots \otimes f_{e^{t_{l-1}}_{t_{l-2}\cdots t_1 k} t_{l-1}^{-1} t_{l-1} t_{l-2} \cdots t_1 k} \otimes f_{e^{t(l-1)}_{t_{l-1}\cdots t_1 k}}.$$

Now we compute

$$
\begin{aligned}
\varDelta^{\mathsf{M}}(t)(\mathbf{v}_k^{t(0)}) &= \varDelta^{\mathsf{M}}(t)(\lambda^k_{e_k}(\mathbf{j}; f_{e_k^{t(0)}}) - \mathbf{j}\rho_\varnothing) = \lambda^k_{e_k}(\mathbf{j}; \varDelta^{\mathsf{M}}(t)(f_{e_k^{t(0)}})) - \mathbf{j}\rho_\varnothing \\
&= \mathbf{j} f_{e_k^{t_1^{-1}t_1k}} \otimes f_{e_{t_1k}^{t_2^{-1}t_2t_1k}} \otimes f_{e_{t_2t_1k}^{t_3^{-1}t_3t_2t_1k}} \otimes \cdots \otimes f_{e_{t_{l-2}\dots t_1k}^{t_{l-1}^{-1}t_{l-1}t_{l-2}\dots t_1k}} \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad - \mathbf{j}\rho_\varnothing \otimes f_{e_{t_1k}^{t_2^{-1}t_2t_1k}} \otimes f_{e_{t_2t_1k}^{t_3^{-1}t_3t_2t_1k}} \otimes \cdots \otimes f_{e_{t_{l-2}\dots t_1k}^{t_{l-1}^{-1}t_{l-1}t_{l-2}\dots t_1k}} \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad + \mathbf{j} f_{e_{t_1k}^{t_2^{-1}t_2t_1k}} \otimes f_{e_{t_2t_1k}^{t_3^{-1}t_3t_2t_1k}} \otimes \cdots \otimes f_{e_{t_{l-2}\dots t_1k}^{t_{l-1}^{-1}t_{l-1}t_{l-2}\dots t_1k}} \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad - \mathbf{j}\rho_\varnothing \otimes f_{e_{t_2t_1k}^{t_3^{-1}t_3t_2t_1k}} \otimes \cdots \otimes f_{e_{t_{l-2}\dots t_1k}^{t_{l-1}^{-1}t_{l-1}t_{l-2}\dots t_1k}} \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad + \mathbf{j} f_{e_{t_2t_1k}^{t_3^{-1}t_3t_2t_1k}} \otimes \cdots \otimes f_{e_{t_{l-2}\dots t_1k}^{t_{l-1}^{-1}t_{l-1}t_{l-2}\dots t_1k}} \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad \cdots \\
&\quad - \mathbf{j}\rho_\varnothing \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad + \mathbf{j} f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad - \mathbf{j}\rho_\varnothing \\
&= \mathbf{v}_k^{t_1^{-1}t_1k} \otimes f_{e_{t_1k}^{t_2^{-1}t_2t_1k}} \otimes f_{e_{t_2t_1k}^{t_3^{-1}t_3t_2t_1k}} \otimes \cdots \otimes f_{e_{t_{l-2}\dots t_1k}^{t_{l-1}^{-1}t_{l-1}t_{l-2}\dots t_1k}} \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad + \mathbf{v}_{t_1k}^{t_2^{-1}t_2t_1k} \otimes f_{e_{t_2t_1k}^{t_3^{-1}t_3t_2t_1k}} \otimes \cdots \otimes f_{e_{t_{l-2}\dots t_1k}^{t_{l-1}^{-1}t_{l-1}t_{l-2}\dots t_1k}} \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad + \mathbf{v}_{t_2t_1k}^{t_3^{-1}t_3t_2t_1k} \otimes \cdots \otimes f_{e_{t_{l-2}\dots t_1k}^{t_{l-1}^{-1}t_{l-1}t_{l-2}\dots t_1k}} \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad \cdots \\
&\quad + \mathbf{v}_{t_{l-2}\dots t_1k}^{t_{l-1}^{-1}t_{l-1}t_{l-2}\dots t_1k} \otimes f_{e_{t_{l-1}\dots t_1k}^{t(l-1)}} \\
&\quad + \mathbf{v}_{t_{l-1}\dots t_1k}^{t(l-1)}.
\end{aligned}
$$

On other generators we transform

$$
\varDelta^{\mathsf{M}}(t)(f_{(\ell_1^k;\ell_2^k;\dots;\ell_{t^k}^k)_{k\in\mathbf{n}}}) = \lambda_{\hat{\ell}}\big(({}^{\ell_1^k}1, \mathbf{j}, {}^{\ell_2^k}1, \mathbf{j}, \dots, {}^{\ell_{t^k-1}^k}1, \mathbf{j}, {}^{\ell_{t^k}^k}1)_{k\in\mathbf{n}}; \varDelta^{\mathsf{M}}(t)(f^+_{\hat{\ell}})\big)
$$

as follows. First factors f_{i_q} are replaced with generators $f_{a_1;a_2;\dots;a_p}$ accordingly with the set of $\mathbf{j}$'s appearing among the arguments of f_{i_q}. The only exception is the case of $\mathbf{j}f_{e_k}$ which is replaced with $\mathbf{v}_k + \mathbf{j}\rho_\varnothing$. In obtained summands all instances of $\mathbf{j}\rho_\varnothing$ are moved to the right as arguments $\mathbf{j}$ of next f_p due to defining the tensor product as a colimit, and this procedure goes on until no $\mathbf{j}\rho_\varnothing$ are left. Notice that the separate term $\mathbf{j}\rho_\varnothing$ can not appear elsewhere but in the expression $\varDelta^{\mathsf{M}}(t)(\mathbf{j}f_{e_k^{t(0)}})$, which is not considered by itself, but only as a summand of $\mathbf{v}_k^{t(0)}$. $\square$

3.18 Corollary. *The collection of operad submodules* $(\mathrm{A}_\infty^{hu}, \mathrm{F}^{hu}) \subset (\mathrm{A}_\infty^{\mathsf{su}}, \mathrm{F}^{\mathsf{su}})\langle \mathsf{i}, \mathsf{j}\rangle$ *is a polymodule subcooperad.*

A. Colimits of algebras over monads

Let $\top : \mathcal{C} \to \mathcal{C}$ be a monad, and let $F : \mathcal{C} \leftrightarrows \mathcal{C}^\top : U$ be the associated adjunction. Assume that $\mathcal{C}$ is cocomplete and $\mathcal{C}^\top$ has coequalizers. The latter condition is satisfied in each of the two following cases:

- $\mathcal{C}$ is a complete, regular, regularly co-well-powered category with coequalizers, and $\top$ is a monad which preserves regular epimorphisms [BW05, Proposition 9.3.8].
- $\mathcal{C}$ has finite colimits and equalizers of arbitrary sets of maps (with the same source and target), and $\top$ is a monad in $\mathcal{C}$ which preserves colimits along countable chains [BW05, Theorem 9.3.9].

When $\mathcal{C}^\top$ has coequalizers, the category $\mathcal{C}^\top$ is cocomplete by a result of Barr and Wells [BW05, Corollary 9.3.3]. Our goal in this section is to reprove this result expressing the colimit in $\mathcal{C}^\top$ through the colimit in $\mathcal{C}$ via sufficiently explicit recipe.

A.1 Proposition. *Assume that* $\mathcal{C}$ *is cocomplete and* $\mathcal{C}^\top$ *has coequalizers. Then the category* $\mathcal{C}^\top$ *is cocomplete.*

Proof. Let I be a small category and let $I \ni i \mapsto P_i \in \mathcal{C}^\top$ be a diagram in $\mathcal{C}^\top$. Denote by C the colimit (coequalizer) of the following diagram in $\mathcal{C}^\top$

$$\begin{tikzcd}
F\operatorname*{colim}_i UFUP_i \arrow[d, "F\operatorname{colim}\alpha_i"'] \arrow[dr, "F\,\mathrm{can}"] & & \\
F\operatorname*{colim}_i UP_i \arrow[d, "F\operatorname{colim}\eta"'] & FUF\operatorname*{colim}_i UP_i \arrow[r, "\varepsilon"] & F\operatorname*{colim}_i UP_i \\
F\operatorname*{colim}_i UFUP_i \arrow[ur, "F\,\mathrm{can}"'] & &
\end{tikzcd} \tag{A.1}$$

where 'can' means any canonical map. Equip $C = (C, \mathrm{can} : F\operatorname{colim}_i UP_i \to C)$ with maps in $\mathcal{C}$ going through the rightmost vertex

$$\begin{aligned}
\mathrm{In}_i &= \bigl(UP_i \xrightarrow{\mathrm{in}_i} \operatorname*{colim}_i UP_i \xrightarrow{\eta} UF\operatorname*{colim}_i UP_i \xrightarrow{U\,\mathrm{can}} UC\bigr) \\
&= \bigl(UP_i \xrightarrow{\eta} UFUP_i \xrightarrow{UF\,\mathrm{in}_i} UF\operatorname*{colim}_i UP_i \xrightarrow{U\,\mathrm{can}} UC\bigr).
\end{aligned} \tag{A.2}$$

We claim that $\mathrm{In}_i \in \mathcal{C}^\top$ and $(C, \mathrm{In}_i : P_i \to C \mid i \in I)$ is the colimiting cocone of the diagram $i \mapsto P_i$ in $\mathcal{C}^\top$.

Let us verify that In_i are morphisms of $\top$-algebras. The exterior of the following diagram commutes

$$\begin{tikzcd} UFUP_i \arrow[r, "UF\eta"] \arrow[d, "\alpha_i"'] \arrow[drr, "UF\,\mathrm{in}_i" description] & UFUFUP_i \arrow[r, "UFUF\,\mathrm{in}_i"] \arrow[d, phantom, "="] & UFUF\operatorname*{colim}_i UP_i \arrow[r, "UFU\,\mathrm{can}"] \arrow[d, "U\varepsilon"] \arrow[dr, phantom, "="] & UFUC \arrow[d, "\alpha_C"] \\ UP_i \arrow[r, "\eta"] & UFUP_i \arrow[r, "UF\,\mathrm{in}_i"] & UF\operatorname*{colim}_i UP_i \arrow[r, "U\,\mathrm{can}"] & UC \end{tikzcd} \tag{A.3}$$

if and only if

$$\begin{tikzcd} UFUP_i \arrow[r, "UF\,\mathrm{in}_i"] \arrow[d, "\alpha_i"'] & UF\operatorname*{colim}_i UP_i \arrow[r, "U\,\mathrm{can}"] \arrow[d, phantom, "="] & UC \\ UP_i \arrow[r, "\eta"] & UFUP_i \arrow[r, "UF\,\mathrm{in}_i"] & UF\operatorname*{colim}_i UP_i \arrow[u, "U\,\mathrm{can}"'] \end{tikzcd} \tag{A.4}$$

Schematically this is the equation $f = \big(A \xrightarrow{g} A \xrightarrow{f} C\big)$, where $f = UF\,\mathrm{in}_i \cdot U\,\mathrm{can} : A \to C \in \mathcal{C}^\top$ but $g = \alpha_i \cdot \eta \in \mathcal{C}$. By the freeness of $\top$-algebra $\top A$ (see the proof of [BW05, Theorem 3.2.1]) this is equivalent to equation

$$\big(\top A \xrightarrow{\alpha_A} A \xrightarrow{f} C\big) = \big(\top A \xrightarrow{\top g} \top A \xrightarrow{\alpha_A} A \xrightarrow{f} C\big).$$

In detail it is the equation

$$\begin{tikzcd} UFUFUP_i \arrow[r, "U\varepsilon"] \arrow[d, "UF\alpha_i"'] & UFUP_i \arrow[r, "UF\,\mathrm{in}_i"] & UF\operatorname*{colim}_i UP_i \arrow[r, "U\,\mathrm{can}"] \arrow[d, phantom, "="] & UC \\ UFUP_i \arrow[r, "UF\eta"] & UFUFUP_i \arrow[r, "U\varepsilon"] & UFUP_i \arrow[r, "UF\,\mathrm{in}_i"] & UF\operatorname*{colim}_i UP_i \arrow[u, "U\,\mathrm{can}"'] \end{tikzcd} \tag{A.5}$$

Removing the unnecessary U we write it as an equation in $\mathcal{C}^\top$:

$$\begin{tikzcd} FUFUP_i \arrow[r, "FUF\,\mathrm{in}_i"] \arrow[d, "F\alpha_i"'] & FUF\operatorname*{colim}_i UP_i \arrow[r, "\varepsilon"] & F\operatorname*{colim}_i UP_i \arrow[r, "\mathrm{can}"] \arrow[d, phantom, "="] & C \\ FUP_i \arrow[r, "F\eta"] & FUFUP_i \arrow[r, "FUF\,\mathrm{in}_i"] & FUF\operatorname*{colim}_i UP_i \arrow[r, "\varepsilon"] & F\operatorname*{colim}_i UP_i \arrow[u, "\mathrm{can}"'] \end{tikzcd} \tag{A.6}$$

which holds due to (C, can) being coequalizer of (A.1).

Clearly, $\mathrm{In}_i : P_i \to C$ is a cocone from the diagram $i \mapsto P_i$. Let us prove that it is an initial one. Let $\phi_i : P_i \to Q \in \mathcal{C}^\top$ be an arbitrary cocone from the diagram $i \mapsto P_i$. There is a unique map $\beta : \mathrm{colim}_i UP_i \to UQ \in \mathcal{C}$ such that $U\phi_i = \big(UP_i \xrightarrow{\mathrm{in}_i} \mathrm{colim}_i UP_i \xrightarrow{\beta} UQ\big)$. It has an adjunct $\gamma = {}^t\beta = \big(F\,\mathrm{colim}_i UP_i \xrightarrow{F\beta} FUQ \xrightarrow{\varepsilon} Q\big) \in \mathcal{C}^\top$, so that ${}^t(U\phi_i) = \big(FUP_i \xrightarrow{F\,\mathrm{in}_i} F\,\mathrm{colim}_i UP_i \xrightarrow{\gamma} Q\big)$. Consequently,

$$U\phi_i = \big(UP_i \xrightarrow{\eta} UFUP_i \xrightarrow{UF\,\mathrm{in}_i} UF\operatorname*{colim}_i UP_i \xrightarrow{U\gamma} UQ\big).$$

Since $\phi_i \in \mathcal{C}^\top$ the exterior of diagram (A.3) commutes, where can and C are replaced with γ and Q. Therefore, equation (A.4) with the same replacement holds. As explained above this implies equations (A.5) and (A.6) with the same modification. Therefore, both paths in diagram (A.1) postcomposed with $\gamma : F\operatorname{colim}_i UP_i \to Q$ from the top vertex $F\operatorname{colim}_i UFUP_i$ to Q are equal to each other. Hence, γ factorizes as $F\operatorname{colim}_i UP_i \xrightarrow{\mathrm{can}} C \xrightarrow{\psi} Q$ for a unique $\psi \in \mathcal{C}^\top$. $\square$

A.2 Remark. It is shown in the proof of the above proposition that $\operatorname{colim}_i P_i$ is the biggest quotient of $F\operatorname{colim}_i UP_i$ via a regular epimorphism $\mathrm{can} : F\operatorname{colim}_i UP_i \to C = \operatorname{colim}_i P_i$ such that morphisms $\mathrm{In}_i : UP_i \to UC \in \mathcal{C}$ from (A.2) are morphisms of $\top$-algebras.

A.3 Proposition. *Assume that $\mathcal{C}$ is cocomplete and $\mathcal{C}^\top$ has coequalizers. Let $X \in \operatorname{Ob}\mathcal{C}$ and $A = (UA, \alpha : UFUA \to UA) \in \operatorname{Ob}\mathcal{C}^\top$. Then the colimit $C = (C, \mathrm{can} : F(X \sqcup UA) \to C)$ of the diagram in $\mathcal{C}^\top$*

$$\begin{tikzcd}
FUFUA \arrow[d, "F\alpha"'] \arrow[drr, "FUF\,\mathrm{in}_2"] & & & \\
FUA \arrow[d, "F\eta"'] & & FUF(X \sqcup UA) \arrow[r, "\varepsilon"] & F(X \sqcup UA) \\
FUFUA \arrow[urr, "FUF\,\mathrm{in}_2"'] & & &
\end{tikzcd} \tag{A.7}$$

equipped with the morphisms of $\top$-algebras

$$\begin{aligned}
\mathrm{In}_1 &= \big(FX \xrightarrow{F\,\mathrm{in}_1} F(X \sqcup UA) \xrightarrow{\mathrm{can}} C\big), \\
\mathrm{In}_2 &= \big(UA \xrightarrow{\mathrm{in}_2} X \sqcup UA \xrightarrow{\eta} UF(X \sqcup UA) \xrightarrow{U\,\mathrm{can}} UC\big) \\
&= \big(UA \xrightarrow{\eta} UFUA \xrightarrow{UF\,\mathrm{in}_2} UF(X \sqcup UA) \xrightarrow{U\,\mathrm{can}} UC\big)
\end{aligned} \tag{A.8}$$

is the coproduct $FX \sqcup A$ in $\mathcal{C}^\top$.

Proof. Let us verify that In_2 is a morphism of $\top$-algebras. This is equivalent to commutativity of the exterior of the following diagram

$$\begin{tikzcd}
UFUA \arrow[r, "UF\eta"] \arrow[d, "\alpha"'] \arrow[drr, "UF\,\mathrm{in}_2"] & UFUFUA \arrow[r, "UFUF\,\mathrm{in}_2"] \arrow[d, phantom, "="] & UFUF(X \sqcup UA) \arrow[r, "UFU\,\mathrm{can}"] \arrow[d, "U\varepsilon"] & UFUC \arrow[d, "\alpha_C"] \arrow[dl, phantom, "="] \\
UA \arrow[r, "\eta"] & UFUA \arrow[r, "UF\,\mathrm{in}_2"] & UF(X \sqcup UA) \arrow[r, "U\,\mathrm{can}"] & UC
\end{tikzcd} \tag{A.9}$$

which holds if and only if

$$\begin{tikzcd}
UFUA \arrow[r, "UF\,\mathrm{in}_2"] \arrow[d, "\alpha"'] & UF(X \sqcup UA) \arrow[r, "U\,\mathrm{can}"] \arrow[d, phantom, "="] & UC \\
UA \arrow[r, "\eta"] & UFUA \arrow[r, "UF\,\mathrm{in}_2"] & UF(X \sqcup UA) \arrow[u, "U\,\mathrm{can}"']
\end{tikzcd} \tag{A.10}$$

Schematically this is the equation $f = \big(B \xrightarrow{g} B \xrightarrow{f} C\big)$, where $f = UF\,\mathrm{in}_2 \cdot U\,\mathrm{can} : B \to C \in \mathcal{C}^\top$ but $g = \alpha \cdot \eta \in \mathcal{C}$. By the freeness of $\top$-algebra $\top B$ (see the proof of [BW05, Theorem 3.2.1]) this is equivalent to equation

$$\big(\top B \xrightarrow{\alpha_B} B \xrightarrow{f} C\big) = \big(\top B \xrightarrow{\top g} \top B \xrightarrow{\alpha_B} B \xrightarrow{f} C\big).$$

In detail it is the equation

$$\begin{tikzcd}
UFUFUA \arrow[r, "U\varepsilon"] \arrow[d, "UF\alpha"'] & UFUA \arrow[r, "UF\,\mathrm{in}_2"] & UF(X \sqcup UA) \arrow[r, "U\,\mathrm{can}"] & UC \\
& & = & \\
UFUA \arrow[r, "UF\eta"] & UFUFUA \arrow[r, "U\varepsilon"] & UFUA \arrow[r, "UF\,\mathrm{in}_2"] & UF(X \sqcup UA) \arrow[uu, "U\,\mathrm{can}"']
\end{tikzcd} \tag{A.11}$$

Removing the unnecessary U we write it as an equation in $\mathcal{C}^\top$:

$$\begin{tikzcd}
FUFUA \arrow[r, "FUF\,\mathrm{in}_2"] \arrow[d, "F\alpha"'] & FUF(X \sqcup UA) \arrow[r, "\varepsilon"] & F(X \sqcup UA) \arrow[r, "\mathrm{can}"] & C \\
& & = & \\
FUA \arrow[r, "F\eta"] & FUFUA \arrow[r, "FUF\,\mathrm{in}_2"] & FUF(X \sqcup UA) \arrow[r, "\varepsilon"] & F(X \sqcup UA) \arrow[uu, "\mathrm{can}"']
\end{tikzcd} \tag{A.12}$$

which holds due to (C, can) being coequalizer of (A.7).

Let us prove that $(C, \mathrm{In}_1 : FX \to C, \mathrm{In}_2 : A \to C)$ is the coproduct $FX \sqcup A$ in $\mathcal{C}^\top$. Let $\phi_1 : FX \to Q \in \mathcal{C}^\top$ and $\phi_2 : A \to Q \in \mathcal{C}^\top$. The maps $\delta = \phi_1^t = \big(X \xrightarrow{\eta} UFX \xrightarrow{U\phi_1} UQ\big)$ and $U\phi_2 : UA \to UQ$ determine a unique map $\beta : X \sqcup UA \to UQ$ in $\mathcal{C}$. It has an adjunct $\gamma = {}^t\beta = \big(F(X \sqcup UA) \xrightarrow{F\beta} FUQ \xrightarrow{\varepsilon} Q\big) \in \mathcal{C}^\top$, so that $\phi_1 = {}^t\delta = \big(FX \xrightarrow{F\,\mathrm{in}_1} F(X \sqcup UA) \xrightarrow{\gamma} Q\big)$, ${}^t(U\phi_2) = \big(FUA \xrightarrow{F\,\mathrm{in}_2} F(X \sqcup UA) \xrightarrow{\gamma} Q\big)$. Consequently,

$$U\phi_2 = \big(UA \xrightarrow{\eta} UFUA \xrightarrow{UF\,\mathrm{in}_2} UF(X \sqcup UA) \xrightarrow{U\gamma} UQ\big).$$

Since $\phi_2 \in \mathcal{C}^\top$ the exterior of diagram (A.9) commutes, where can and C are replaced with γ and Q. Therefore, equation (A.10) with the same replacement holds. As explained above this implies equations (A.11) and (A.12) with the same modification. Therefore, both paths in diagram (A.7) postcomposed with $\gamma : F(X \sqcup UA) \to Q$ from the top vertex $FUFUA$ to Q are equal to each other. Hence, γ factorizes as $F(X \sqcup UA) \xrightarrow{\mathrm{can}} C \xrightarrow{\psi} Q$ for a unique $\psi \in \mathcal{C}^\top$. We get

$$\phi_1 = \big(FX \xrightarrow{F\,\mathrm{in}_1} F(X \sqcup UA) \xrightarrow{\mathrm{can}} C \xrightarrow{\psi} Q\big) = \big(FX \xrightarrow{\mathrm{In}_1} C \xrightarrow{\psi} Q\big),$$
$$U\phi_2 = \big(UA \xrightarrow{\eta} UFUA \xrightarrow{UF\,\mathrm{in}_2} UF(X \sqcup UA) \xrightarrow{U\,\mathrm{can}} UC \xrightarrow{U\psi} UQ\big),$$

hence, $\phi_2 = \big(A \xrightarrow{\mathrm{In}_2} C \xrightarrow{\psi} Q\big)$. This shows that $(C, \mathrm{In}_1, \mathrm{In}_2)$ is the coproduct $FX \sqcup A$ in $\mathcal{C}^\top$. □

A.4 Corollary. *$FX \sqcup A$ is the biggest quotient of $F(X \sqcup UA)$ via a regular epimorphism* $\mathrm{can} : F(X \sqcup UA) \to C = FX \sqcup A$ *such that the morphism* $\mathrm{In}_2 : UA \to UC \in \mathcal{C}$ *from* (A.8) *is a morphism of $\top$-algebras.*

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